

Problem 1. [Upper bound of the lowest eigenvalue by using Rayleigh quotient]

In Problem 6 of Homework 10 you found the eigenvalues and eigenfunctions of a Sturm-Liouville problem with Neumann/Robin BCs by a change of variables which transformed the ODE to the harmonic oscillator ODE. Now you will consider the same problem (with concrete numerical values of the constants) and will find an upper bound on the lowest eigenvalue, λ_1 , by using the Rayleigh quotient. The Sturm-Liouville problem is

$$\begin{aligned} \frac{d}{dx} \left(x \frac{dy}{dx} \right) + \frac{\lambda}{x} y(x) &= 0, & x \in [1, 2], \\ y'(1) &= 0, & y'(2) + y(2) = 0. \end{aligned} \quad (1)$$

- (a) Assume that $\phi(x)$ is an eigenfunction of the Sturm-Liouville problem (1) with eigenvalue λ , i.e., $\phi(x)$ satisfies (1). Multiply the ODE for $\phi(x)$ by $\phi(x)$, and integrate the resulting equality with respect to x over the interval $[1, 2]$. From the equality you obtain, express the eigenvalue λ .
- (b) Integrate by parts to prove the identity

$$\int_1^2 \phi(x) [x\phi'(x)]' dx = 2\phi(2)\phi'(2) - \phi(1)\phi'(1) - \int_1^2 x \phi'(x)^2 dx. \quad (2)$$

Please write your calculations in detail.

- (c) Use the boundary conditions from (1) to rewrite the boundary terms in (2) (i.e., the terms containing the values of ϕ and its derivative at the boundary points 1 and 2). Use this and the result from part (b) to rewrite the expression for λ from part (a).
- (d) If you take the minimum of the Rayleigh quotient obtained in part (c) over all functions $\phi(x)$ that satisfy the boundary conditions from (1), you will obtain the exact value of the lowest eigenvalue λ_1 . If you take any function $\psi(x)$ that satisfies the boundary conditions from (1) and substitute it in the Rayleigh quotient, you will obtain a rigorous upper bound on λ_1 . Take as a trial function (which will be used in the Rayleigh quotient) a function from the 3-parameter family

$$\psi_{a,b,c}(x) = a + bx + cx^2.$$

Impose the BCs from (1) to find two relations between the three parameters a , b , and c . Take $c = -1$ and the values of a and b coming from the relations you just derived. You will obtain that the trial function is

$$\psi(x) := \psi_{2,2,-1}(x) = 2 + 2x - x^2,$$

but I want to see your detailed calculations.

- (e) Plug the function $\psi(x)$ from part (d) into the Rayleigh quotient to obtain a rigorous upper bound on λ_1 . You may use the integrals

$$\int_1^2 \frac{1}{x} (2 + 2x - x^2)^2 dx = \frac{29}{12} + \ln 16 , \quad \int_1^2 x (2 - 2x)^2 dx = \frac{7}{3} .$$

Compute the numerical value of the upper bound with accuracy of 6 or more digits.

- (f) As obtained in Problem 6 of Homework 10, the eigenvalues of the Sturm-Liouville eigenvalue problem (1) are $\lambda_n = \alpha_n^2$, where α_n is the n th positive root of the equation

$$\tan(\alpha \ln 2) = \frac{2}{\alpha} , \quad (3)$$

and the eigenfunctions are $y_n(x) = \cos(\alpha_n \ln x)$. The numerical value of α_1 computed from (3) is $\alpha_1 = 1.38996056924145 \dots$. Compare the corresponding value of $\lambda_1 = \alpha_1^2$ with the rigorous upper bound on λ_1 obtained in part (e). Compute the relative error,

$$\frac{|\lambda_1^{(\text{upper bound})} - \lambda_1^{(\text{exact})}|}{|\lambda_1^{(\text{exact})}|} \times 100 \% .$$

Remark: Although the exact eigenvalue $\lambda_1^{(\text{exact})}$ and the upper bound $\lambda_1^{(\text{upper bound})}$ are close, the true eigenfunction $y_1(x) = \cos(x \ln 2)$ and the trial function, $\psi(x) = \frac{1}{3}(2 + 2x - x^2)$ (normalized so that $\psi(1) = y_1(x)$), may be quite different. These two functions are shown in Figure 1.

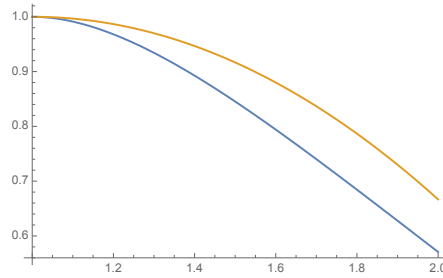


Figure 1: The trial function $\psi(x) = \frac{1}{3}(2 + 2x - x^2)$ (on top) and the true eigenfunction $y_1(x)$ of the Sturm-Liouville eigenvalue problem (1).

Problem 2. [Hopf-Cole transformation of a nonlinear equation]

In this problem you will solve the IBVP

$$\begin{aligned} u_t &= u_{xx} + (u_x)^2 + e^{-u} \sin x , & (x, t) &\in [0, \pi] \times \mathbb{R}_+ , \\ u(0, t) &= 0 , \quad u(\pi, t) = 0 , \\ u(x, 0) &= 0 , \end{aligned} \quad (4)$$

where $u = u(x, t)$ is a function of two variables. The difficulty in solving this IBVP is that the partial differential equation in (4) is nonlinear (due to the presence of the terms $(u_x)^2$ and e^{-u}) and, hence, cannot be solved with the standard methods learned in this class. It, however, can be transformed to a linear equation by the so-called *Hopf-Cole transformation*,

$$u(x, t) = \ln v(x, t) , \quad (5)$$

where $v(x, t)$ is a new unknown function.

One can express the derivatives of the original function u in terms of the new function v and its derivatives. For example,

$$u_t(x, t) = \frac{\partial}{\partial t} u(x, t) = \frac{\partial}{\partial t} \ln v(x, t) = \frac{1}{v(x, t)} \frac{\partial}{\partial t} v(x, t) = \frac{v_t(x, t)}{v(x, t)} ,$$

where we have used the Chain Rule and the fact that $\frac{d}{dz} \ln z = \frac{1}{z}$.

You have already solved a part of this problem in Problem 1 of Homework 1.

- (a) Use the Chain Rule to express $u_x(x, t)$ in terms of $v(x, t)$ and its derivatives.
- (b) Use the Chain Rule again to find $u_{xx}(x, t)$ in terms of $v(x, t)$ and its derivatives.
- (c) Use your results from parts (a) and (b) to show that the Hopf-Cole transformation (5) transforms the nonlinear equation (4) into a simpler equation (which does not contain nonlinear terms like the ones in (4)).
- (d) Transform the boundary conditions and the initial conditions for the function $u(x, t)$ from the IBVP (4) to boundary and initial conditions for the function $v(x, t)$ to obtain the IBVP

$$\begin{aligned} v_t &= v_{xx} + \sin x , & (x, t) &\in [0, \pi] \times \mathbb{R}_+ , \\ v(0, t) &= 1 , \quad v(\pi, t) = 1 , \\ v(x, 0) &= 1 . \end{aligned} \quad (6)$$

Since the boundary conditions for v are non-homogeneous (i.e., non-zero), we can set

$$v(x, t) = 1 + w(x, t)$$

and transform the IBVP (6) for $v(x, t)$ into the following IBVP for $w(x, t)$:

$$\begin{aligned} w_t &= w_{xx} + \sin x , & (x, t) &\in [0, \pi] \times \mathbb{R}_+ , \\ w(0, t) &= 0 , \quad w(\pi, t) = 0 , \\ w(x, 0) &= 0 . \end{aligned} \quad (7)$$

- (e) The partial differential equation in the IBVP (7) is the heat equation with heat sources – the term $\sin x$ in the right-hand side of the PDE corresponds to sources of heat distributed in the interval $[0, \pi]$ with linear density proportional to $\sin x$.

We learned how to deal with source terms in Problem 2 of Homework 7. The idea was the following. If there were no sources of heat in (7), then separation of variables will lead to the BVP

$$\begin{aligned} X''(x) - \mu X &= 0, \quad x \in [0, \pi], \\ X(0) &= 0, \quad X(\pi) = 0, \end{aligned} \tag{8}$$

whose non-trivial solutions are

$$X_n(x) = \sin \lambda_n x = \sin nx, \quad \lambda_n = n, \quad n = 1, 2, 3, \dots,$$

and the corresponding values of the constant μ from the separation of variables are $\mu_n = -\lambda_n^2 = -n^2$. Therefore, the solution $w(x, t)$ (without source of heat!) would be a superposition of $X_n(x) = \sin nx$ with coefficients $T_n(t)$:

$$w(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin nx. \tag{9}$$

This expression satisfies the BCs in (7), but not the PDE or the initial conditions. Nevertheless, we will try to find a solution for the IBVP (7) as an expansion of the form (9). We have to find functions $T_n(t)$ to satisfy also the PDE and the initial conditions. We expand the source function $Q(x, t) = \sin x$ from the right-hand side of the PDE from (7) in the system of eigenfunctions $X_n(x) = \sin nx$ from (8):

$$Q(x, t) = \sum_{n=1}^{\infty} Q_n(t) \sin nx; \tag{10}$$

clearly, for $Q(x, t) = \sin x$ we have $Q_n(t) = \delta_{1n}$, where δ_{ij} is the Kronecker's symbol.

Plug the expansions (9) for $w(x, t)$ and (10) for $Q(x, t)$ into the PDE from (7), and use the fact that the functions $X_n(x)$ are linearly independent, to equate the coefficients and write down ODEs that the functions $T_n(t)$ must satisfy.

- (f) Set $t = 0$ in (9) and use the initial condition from (7) to find initial conditions for the ODEs for $T_n(t)$ from (9) (in other words, find the values of $T_n(0)$).
- (g) The only non-trivial $T_n(t)$ is $T_1(t)$ (think why!). Write down and solve the initial value problem for the function $T_1(t)$. (The simple trick from the solution of Problem 2(d) of Homework 7 will be useful.)
- (h) Plug your result for $T_1(t)$ from part (g) into (9) to find $w(x, t)$, then find $v(x, t)$, and finally find the solution $u(x, t)$ of the original IBVP (4).