MATH 3423 Homework 11 Due Thu, 12/5/19

Problem 1. Use the relations $\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ and $\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$ (which are obvious from the definitions of $\mathbf{e}_r$ and $\mathbf{e}_\theta$) to show that $\dfrac{\mathrm{d}\mathbf{e}_\theta}{\mathrm{d}\theta} = -\mathbf{e}_r$.

Problem 2. Use the relations $\dfrac{\mathrm{d}\mathbf{e}_r}{\mathrm{d}\theta} = \mathbf{e}_\theta$ and $\dfrac{\mathrm{d}\mathbf{e}_\theta}{\mathrm{d}\theta} = -\mathbf{e}_r$ to determine $\dfrac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}$ if

$$\mathbf{u}(t) = \sin\theta(t)\,\mathbf{e}_r - r^2(t)\,\theta(t)\,\mathbf{e}_\theta\ , \qquad \text{where} \quad r(t) = t^2\ , \quad \theta(t) = 2t\ .$$

Problem 3.

(a) Use the relations

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}\ , \qquad \mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$$

to express $\mathbf{i}$ and $\mathbf{j}$ in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

(b) Let $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$. Find $\mathbf{v}(t) := \mathbf{r}'(t)$ in Cartesian coordinates.

(c) Use the identity $1 + \tan^2\theta = \dfrac{1}{\cos^2\theta}$ to express $\cos\theta$ and $\sin\theta$ as functions of $\tan\theta$ only. Assume that $\theta \in [0, \frac{\pi}{2})$.

(d) Use your results from parts (a), (b) and (c) to express $\mathbf{v}(t)$ in polar coordinates for $t > 0$. You will need to express $\sin\theta(t)$ and $\cos\theta(t)$ as functions of t only. To this end, you can use that, for $t > 0$, $\theta(t) = \arctan\dfrac{y(t)}{x(t)} = \arctan\dfrac{t^2}{t} = \arctan t$.

Problem 4.

(a) Let the function f be defined in the first quadrant by $f(x, y) = x^2 + y^2 + \tan\dfrac{y}{x}$. Find the gradient of the function $f(x, y)$ in Cartesian coordinates.

(b) Express the function $f(x, y)$ in polar coordinates. In other words, let

$$x = X(r, \theta) = r\cos\theta\ , \qquad y = Y(r, \theta) = r\sin\theta$$

be the change of variables from Cartesian to polar coordinates, and

$$\tilde{f}(r, \theta) := f\big(X(r, \theta), Y(r, \theta)\big)$$

be the same function expressed in polar coordinates. Find the explicit expression for the function $\tilde{f}(r, \theta)$.

(c) Use the formula for the gradient derived in class to compute the gradient of the function $\tilde{f}(r, \theta)$ in polar coordinates (i.e., as a vector expressed in the basis $\mathbf{e}_r$, $\mathbf{e}_\theta$).

(d) Finally, use the expressions for the basis vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ in polar coordinates in terms of the basis vectors $\mathbf{i}$, $\mathbf{j}$ in order to express the vector obtained in part (c) in this problem in terms of the vectors $\mathbf{i}$ and $\mathbf{j}$, and express the coefficients in terms of Cartesian coordinates. What do you observe?

Problem 5. In this problem you will compute $\exp(\underline{\underline{A}}t)$ for

$$\underline{\underline{A}} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} ,$$

where λ is some real number. Of course, $\exp(\underline{\underline{A}}t)$ can be found directly from the definition of exponent of a matrix, but here you will do it by solving a system of a differential equations.

(a) Let $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ be a function from $\mathbb{R}$ to $\mathbb{R}^2$ that satisfies the initial value problem

$$\mathbf{x}'(t) = \underline{\underline{A}}\,\mathbf{x}(t) , \qquad \mathbf{x}(0) = \mathbf{x}^{(0)} ,$$

where $\underline{\underline{A}}$ is the matrix written above. Write explicitly this initial value problem, without using matrix notations.

(b) Find the general solution of the system of ODEs written in part (a) without using any linear algebra method. The second equation in the system is very simple (it does not contain the function $x_1(t)$). After you find the general solution of the second equation, plug it in the first equation, and solve the first equation for the function $x_1(t)$. Your answer should contain two arbitrary constants.

(c) Impose the initial conditions to obtain the particular solution of the initial value problem written above.

(d) Now use that the solution of the initial value problem can be written as $\mathbf{x}(t) = \exp(\underline{\underline{A}}t)\,\mathbf{x}(0)$ to write down the matrix $\exp(\underline{\underline{A}}t)$.

Problem 6. In this problem you will prove some properties of a particular type of orthogonal coordinate system in $\mathbf{R}^2$, the *parabolic coordinates*. The parabolic coordinates (σ, τ) are related to the Cartesian coordinates (x, y) by

$$x = \sigma\tau \tag{1}$$
$$y = \tfrac{1}{2}(\tau^2 - \sigma^2) , \tag{2}$$

where $\sigma \in \mathbb{R}$, $\tau \in [0, \infty)$.

(a) Expressing $\tau = \frac{x}{\sigma}$ from (1) and substituting it in (2), we see that the surfaces (in the case of two dimensions, lines) $\sigma = \text{const}$ are the parabolas

$$y = \frac{x^2}{2\sigma^2} - \frac{\sigma^2}{2} \ .$$

Find the equation of the lines $\tau = \text{const}$. Identify the families $\sigma = \text{const}$ and $\tau = \text{const}$ in Figure 1.

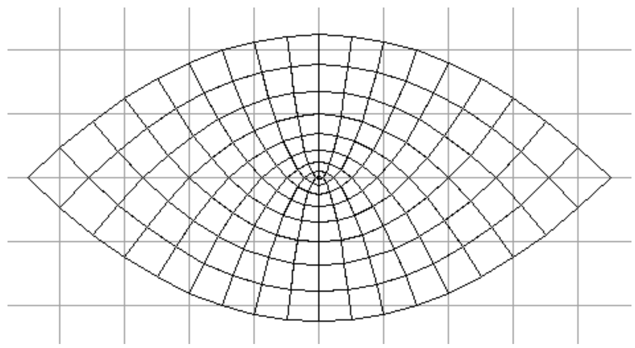

Figure 1: Coordinate lines $\sigma = \text{const}$ and $\tau = \text{const}$.

(b) Let $\mathbf{e}_\sigma$ and $\mathbf{e}_\tau$ be the unit vectors in the coordinates (σ, τ). If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, use (1) and (2) to show that

$$\frac{\partial \mathbf{r}}{\partial \sigma} = \tau \mathbf{i} - \sigma \mathbf{j} \tag{3}$$

$$\frac{\partial \mathbf{r}}{\partial \tau} = \sigma \mathbf{i} + \tau \mathbf{j} \ . \tag{4}$$

(c) The *metric tensor* (g_{ij}) in coordinates $(u_1, \ldots, u_n)$ is given by

$$g_{ij} = \frac{\partial \mathbf{r}}{\partial u_i} \cdot \frac{\partial \mathbf{r}}{\partial u_j} \ .$$

In this problem, it is

$$(g_{ij}) = \begin{pmatrix} g_{\sigma\sigma} & g_{\sigma\tau} \\ g_{\tau\sigma} & g_{\tau\tau} \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathbf{r}}{\partial \sigma} \cdot \frac{\partial \mathbf{r}}{\partial \sigma} & \frac{\partial \mathbf{r}}{\partial \sigma} \cdot \frac{\partial \mathbf{r}}{\partial \tau} \\ \frac{\partial \mathbf{r}}{\partial \tau} \cdot \frac{\partial \mathbf{r}}{\partial \sigma} & \frac{\partial \mathbf{r}}{\partial \tau} \cdot \frac{\partial \mathbf{r}}{\partial \tau} \end{pmatrix} \ .$$

Find the explicit expression of the metric tensor in parabolic coordinates.

(d) From the components of the metric tensor, one can immediately see that (σ, τ) are orthogonal coordinates. How?

(e) Find the *metric coefficients* h_σ and h_τ.

Hint: The metric coefficients for an orthogonal coordinate system are defined by equation (13) on page 357 of the book. However, you have already done all the calculations – the metric coefficients are directly related to the components of the metric tensor.

(f) If the metric tensor in coordinates $(u_1, \ldots, u_n)$ is (g_{ij}), the n-dimensional volume element in these coordinates is given by

$$\mathrm{d}V = \sqrt{\det(g_{ij})}\,\mathrm{d}u_1 \cdots \mathrm{d}u_n \ .$$

In two dimensions, the "volume" is the area, so that the area element $\mathrm{d}A$ in coordinates (u_1, u_2) is given by

$$\mathrm{d}A = \sqrt{\det(g_{ij})}\,\mathrm{d}u_1\,\mathrm{d}u_2 \ . \tag{5}$$

Show that, in parabolic coordinates (σ, τ), this equation becomes

$$\mathrm{d}A = (\sigma^2 + \tau^2)\,\mathrm{d}\sigma\,\mathrm{d}\tau \ .$$

(g) Now consider the domain $\mathscr{D}$ in the (x, y) plane between the lines $x = 0$, $y = 0$, and $y = 1 - \frac{x^2}{4}$, shown in Figure 2. Find the area A of this domain by using each of the

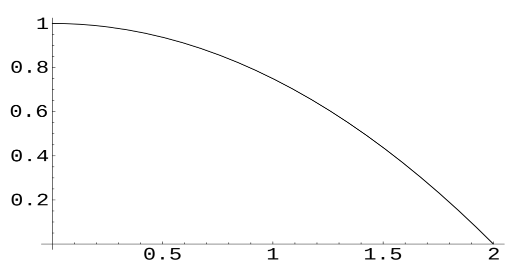

Figure 2: The domain $\mathscr{D}$ in the (x, y) plane.

following three methods (in all these methods use Cartesian coordinates); clearly, each method should give you the same number.

– Use the formula

$$A = \iint \mathrm{d}x\,\mathrm{d}y \ ,$$

where the integration is over the domain

$$\mathscr{D} = \left\{ (x, y) \in \mathbb{R}^2 \ : \ x \geq 0\,,\ y \geq 0\,,\ y \leq 1 - \frac{x^2}{4} \right\} \ .$$

Note that this formula for the area is a particular case of (5) because the metric tensor in Cartesian coordinates is the unit matrix ($g_{ij} = \delta_{ij}$), hence $\det(g_{ij}) = 1$.

– Compute the area A as the area under the graph of the curve $y = Y(x) := 1 - \frac{x^2}{4}$:

$$A = \int_0^2 \left(1 - \frac{x^2}{4}\right) \mathrm{d}x \ .$$

– Now consider y as an independent variable, and $x = X(y)$ as a function, and compute the area A as

$$A = \int_0^{?} X(y)\, \mathrm{d}y \ .$$

(What is the upper limit of integration?)

(h) Now draw the domain $\mathscr{D}$ defined above in the (σ, τ) plane. For your convenience, the domain $\mathscr{D}$ is drawn in coordinates (σ, τ) in the figure below (σ is on the horizontal axis, τ is on the vertical axis). The segment of the y-axis between the points $(x, y) = (0, 0)$

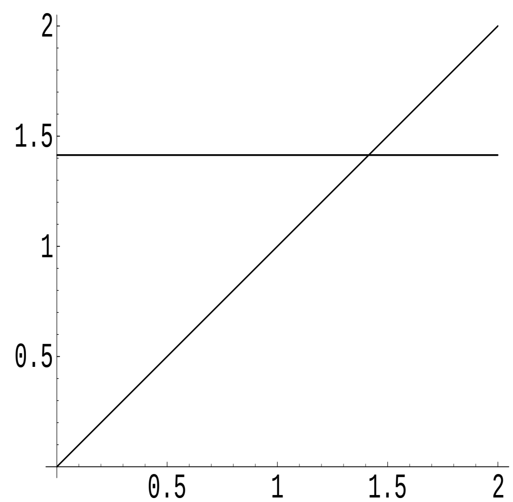

Figure 3: The domain $\mathscr{D}$ in the (σ, τ) plane.

and $(x, y) = (0, 1)$ in Figure 2, is again a straight-line segment in the (σ, τ) plane (Figure 3), connecting the points $(\sigma, \tau) = (0, 0)$ and $(\sigma, \tau) = (0, \sqrt{2})$. You have to write down the equations of the other two parts of the boundary of $\mathscr{D}$ in Figure 2 and to indicate them in Figure 3.

Hint: The parabola in Figure 2 is one of the lines $\tau = \text{const}$; what is the value of the constant?

(i) **[Food for Thought only!]**

Find the area of the domain $\mathscr{D}$ as seen in coordinates (σ, τ), i.e., compute

$$\iint (\sigma^2 + \tau^2)\, \mathrm{d}\sigma\, \mathrm{d}\tau \ ,$$

where the integration is over the domain drawn in Figure 3. You have already computed this area using coordinates (x, y), but I want to see your detailed computations in coordinates (σ, τ).

(j) Follow the ideas from Section 8.2 of the book to find an expression for the gradient of a scalar function $f(\sigma, \tau)$ in parabolic coordinates. Namely, write

$$\mathrm{d}\mathbf{r} = h_\sigma \, \mathrm{d}\sigma \, \mathbf{e}_\sigma + h_\tau \, \mathrm{d}\tau \, \mathbf{e}_\tau \ , \tag{6}$$

$$\nabla f = (\nabla f)_\sigma \, \mathbf{e}_\sigma + (\nabla f)_\tau \, \mathbf{e}_\tau \ ,$$

and use that

$$\mathrm{d}f = (\nabla f) \cdot \mathrm{d}\mathbf{r} = \frac{\partial f}{\partial \sigma} \, \mathrm{d}\sigma + \frac{\partial f}{\partial \tau} \, \mathrm{d}\tau$$

to find expressions for ∇f and the differential operator ∇ (similarly to the equations (22) and (23) on page 359 of the book).

(k) **[Food for Thought only!]**

Finally, you have to compute the length L of the straight-line segment connecting the points $(x, y) = (0, 0)$ and $(x, y) = (2, 0)$ in Figure 2, but using coordinates (σ, τ). Clearly, you have to obtain that $L = 2$. To find L, identify the line segment in the (σ, τ) plane (i.e., in Figure 3), and parametrize it, i.e., write the points in it in the form $(\sigma, \tau) = (\Sigma(t), \mathrm{T}(t))$, where the real parameter t. Since the length element $\mathrm{d}s = |\mathrm{d}\mathbf{r}|$ is given by

$$\mathrm{d}s = \sqrt{h_\sigma^2 \, \mathrm{d}\sigma^2 + h_\tau^2 \, \mathrm{d}\tau^2}$$

(which is easily obtained by squaring (6)), the length L can be obtained as

$$L = \int \mathrm{d}s = \int_?^? \left[h_\sigma^2 \left(\frac{\mathrm{d}\Sigma}{\mathrm{d}t} \right)^2 + h_\tau^2 \left(\frac{\mathrm{dT}}{\mathrm{d}t} \right)^2 \right]^{1/2} \mathrm{d}t$$

(where you have to figure out the limits of integration; clearly, they will depend on the way you parametrized the line segment).