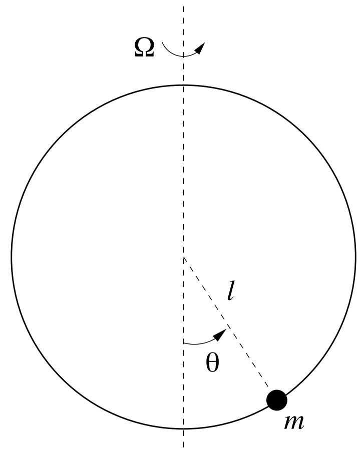
MATH 3423 Homework 11 Due Fri, 12/6/13

Problem 1. A particle with mass m slides without friction around the circumference of a circular wire hoop of radius a. The hoop is placed upright in a uniform gravitational field $\mathbf{g} = -g\mathbf{k}$ and rotates about a vertical diameter with angular velocity $\mathbf{\Omega} = \Omega\mathbf{k}$.



(a) Construct the Lagrangian for the particle using the angle θ (angular displacement measured from the downward vertical) as a generalized coordinate. Work in the laboratory coordinate system (*not* in the coordinate system co-rotating with the hoop). The velocity can be computed by using the expression for the infinitesimal displacement vector $\mathrm{d}\mathbf{r}$ in spherical coordinates (see page 372 of the book):

$$\mathrm{d}\mathbf{r} = \mathrm{d}r\,\mathbf{e}_r + r\,\mathrm{d}\theta\,\mathbf{e}_\theta + r\sin\theta\,\mathrm{d}\phi\,\mathbf{e}_\phi\ ,$$

where r is the distance from the particle to the center of the hoop (*not* to the axis of rotation!); note also that McQuarrie uses notations for the angles different from the ones used in Stewart's Calculus book (see page 369 of McQuarrie for a picture clarifying the meaning of the notations). Dividing by $\mathrm{d}t$, we obtain

$$\dot{\mathbf{r}} = \dot{r}\,\mathbf{e}_r + r\,\dot{\theta}\,\mathbf{e}_\theta + r\sin\theta\,\dot{\phi}\,\mathbf{e}_\phi\ ,$$

so that

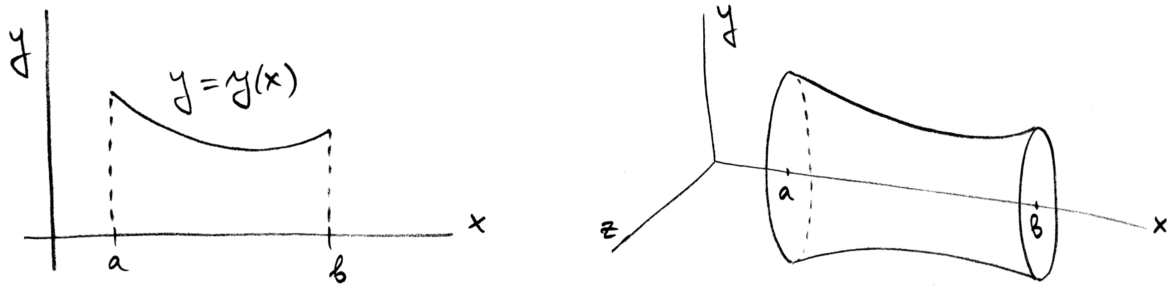
$$|\dot{\mathbf{r}}|^2 = \dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\ . \qquad (1)$$

The distance between the particle and the center of the hoop does not change, and the change of ϕ with time is caused by the uniform rotation of the hoop – these facts should help you write (1) for the system considered here.

(b) Derive the Euler-Lagrange equation for the system.

(c) Write down the Euler-Lagrange equation in the form $\ddot{\theta} = \cdots$, and show that $\theta = 0$ is always an equilibrium position for the particle, and, if $\Omega > \sqrt{\frac{g}{a}}$, then there is another equilibrium position – where is it?

Problem 2. Consider the surface of revolution obtained by rotating a curve $y = y(x)$ in the (x, y)-plane around the x-axis, for $x \in [a, b]$, as shown in the figure below.



Let S stand for the area swept by the curve (note that we do only consider the area of the "cylindrical" part of the figure in the picture on the right, not the areas of the two flat circles at $x = a$ and $x = b$).

The goal in this problem is to find the curve $y = y(x)$ that produces a surface of revolution with the smallest possible area if the two endpoints $(a, y(a))$ and $(b, y(b))$ in the figure on the left are given. This problem has a simple physical interpretation – a soap film whose ends are attached to the two "hoops" at $x = a$ and at $x = b$ in the figure on the right will take exactly the shape that minimizes the surface area because of the surface tension (in this problem we neglect the effect of gravity).

(a) Derive the expression for the area S of the surface of revolution. This expression is a functional of the function y of the form

$$S[y] = \int_a^b L(y, y', x)\,\mathrm{d}x\ , \qquad y' := \frac{\mathrm{d}y}{\mathrm{d}x}\ . \tag{2}$$

Hint: You have solved this problem in Calculus.

(b) Write down the Euler-Lagrange equations for the functional S written in (2).

(c) Note that the function $L(y, y', x)$ does not depend explicitly on x, which according to Problem 1 in Homework 10 implies that the quantity $y' \dfrac{\partial L}{\partial y'} - L$ should be a constant; set

$$y' \frac{\partial L}{\partial y'} - L = \text{const} = 2\pi C_1\ . \tag{3}$$

Write (3) explicitly and show that it implies that

$$y' = \pm \frac{1}{C_1}\sqrt{y^2 - C_1^2}\ ; \tag{4}$$

assume that $C_1 \neq 0$ (because $C_1 = 0$ simply imply that $y(x) \equiv 0$).

(d) Show that the general solution of the ODE (4) can be written in the form

$$y(x) = C_1 \cosh \frac{x - C_2}{C_1} \ . \tag{5}$$

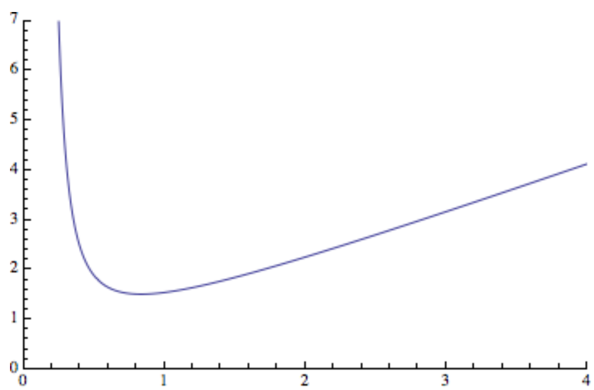
Hint: The integral $\int \frac{\mathrm{d}y}{\sqrt{y^2 - C_1^2}}$ can be solved by using the substitution $y = C_1 \cosh \xi$ and recalling that $\cosh^2 \xi - \sinh^2 \xi = 1$ and $(\cosh \xi)' = \sinh \xi$.

(e) **[Food for thought only, not to be turned in!]**

Now let the initial and the final values of x be $a = -1$ and $b = 1$, and impose the boundary conditions $y(-1) = \beta = y(1)$. From the left-right symmetry of the problem, it is clear that $y(x)$ must be an even function, and since cosh is an even function with a single minimum, it is clear from (5) that the solution $y(x)$ must have the form $y(x) = C_1 \cosh \frac{x}{C_1}$. Imposing the remaining condition,

$$\beta = y(1) = C_1 \cosh \frac{1}{C_1} \ , \tag{6}$$

however, poses a problem: this equation for C_1 has no solution for β in a certain range because the right-hand side of (6) (as a function of C_1) is shown in the figure.



Clearly, if β is smaller than the minimum value of the right-hand side (which is about 1.5, achieved for C_1 approximately equal to 0.83), then there is no value of C_1 that satisfies (6). The absence of solution is due to the fact that the function $y(x)$ that minimizes the area S is not continuous – namely, the function minimizing S is the discontinuous function

$$y(x) = \begin{cases} \beta & x = \pm 1 \ , \\ 0 & x \in (-1, 1) \ . \end{cases}$$

For more on this, see, e.g.,

H. Sagan. *Introduction to the Calculus of Variations.*
Dover Publications, 1992, Section 2.6.

Problem 3. The motion of a membrane in a viscous fluid is governed by the equation

$$\rho\, u_{tt} = \tau\, \Delta u - \gamma u_t + f \ . \tag{7}$$

The notations have the following meaning:

- $z = u(x, y, t)$ is the function describing the position of the membrane at time t;
- ρ is the area density of the mass of the membrane (unit $\mathrm{kg/m^2}$);
- τ is the surface tension (unit $\mathrm{kg/s^2}$),
- γ is the coefficient of resistance (unit $\mathrm{kg/(m^2 s)}$)
- $f(x, y, t)$ is the area density of the external forces (i.e., force per unit area of the membrane; for example, the gravity force will give $f = -\rho g$).

This system is dissipative (because of the term containing the velocity u_t) and cannot be described by a Lagrangian directly. However, it can be derived from the Lagrangian density

$$\mathscr{L}(u, u_x, u_y, u_t, x, y, t) = \left[\frac{\rho}{2}\, u_t^2 - \frac{\tau}{2}\, |\nabla u|^2 + f(x, y, t)\, u\right] \mathrm{e}^{\frac{\gamma}{\rho}\, t} \ . \tag{8}$$

Perform the calculations to derive (7) as the Euler-Lagrange equation corresponding to (8).

A remark about notation: For point particles whose position is described by the generalized coordinates $(q_1(t), \ldots, q_K(t))$, the Lagrangian is a function

$$L := L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) = L(q_1(t), \ldots, q_K(t), \dot{q}_1(t), \ldots, \dot{q}_K(t), t) \ ,$$

and the action has the form

$$I[q] = \int_{t_1}^{t_2} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)\, \mathrm{d}t \ . \tag{9}$$

Let u be a field depending on the spatial coordinate(s) $\mathbf{r}$ and on time t; here $\mathbf{r}$ stand for x or for (x, y) or for (x, y, z), or...; let us assume that $\mathbf{r} \in \mathbb{R}^3$. The action for the field $u(\mathbf{r}, t)$ is given by

$$I[u] = \int_{t_1}^{t_2} \iiint_D \mathscr{L}\big(u, \nabla u, u_t, \mathbf{r}, t\big)\, \mathrm{d}\mathbf{r}\, \mathrm{d}t \ ,$$

where D is a given domain in $\mathbb{R}^3$, and the function $\mathscr{L}$ is called *Lagrangian density* for the simple reason that we integrate not only with respect to time as in (9), but also with respect to the spatial coordinates. The Euler-Lagrange equation in this case is

$$\frac{\partial \mathscr{L}}{\partial u} - \nabla \cdot \left(\frac{\partial \mathscr{L}}{\partial\, \nabla u}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial u_t}\right) = 0 \ ,$$

i.e., $\displaystyle \frac{\partial \mathscr{L}}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial \mathscr{L}}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \mathscr{L}}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \mathscr{L}}{\partial u_z}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial u_t}\right) = 0.$