MATH 3413 **Homework 11** **Due Thu, May 3, 2012**

Problem 1.

(a) Find the general solution of the partial differential equation

$$u_{xyy}(x,y) = 2x\sin y$$

in three different ways as follows:

- first integrate with respect to x and then integrate twice with respect to y;
- first integrate with respect to y, then with respect to x, and then again with respect to y;
- first integrate twice with respect to y and then integrate with respect to x.

(b) Discuss your results from part (a): Did you get the same result by integrating the equation in different order? How many arbitrary functions are in the general solution, and on how many variables does each of these arbitrary functions depend? Is this what you expected?

Problem 2. Consider the wave equation for the function $u(x,t)$ of one spatial (x) and one temporal (t) variables:

$$u_{tt}(x,t) = c^2\, u_{xx}(x,t)\ , \qquad x\in\mathbb{R}\,,\ t\in[0,\infty)\ ,$$

where c is the speed of the wave (measured in meters per second).

(a) Let $\sigma = \Sigma(x,t)$ and $\gamma = \Gamma(x,t)$ be new variables defined as

$$\sigma = \Sigma(x,t) := x - ct\,, \quad \gamma = \Gamma(x,t) := x + ct\ .$$

Let $\widetilde{u}(\sigma,\gamma)$ be a function of two variables defined as

$$u(x,t) := \widetilde{u}\big(\Sigma(x,t),\Gamma(x,t)\big) = \widetilde{u}(\sigma,\gamma)|_{\sigma=\Sigma(x,t),\ \gamma=\Gamma(x,t)}$$

Using the standard jargon, $\widetilde{u}(\sigma,\gamma)$ is the function $u(x,t)$ expressed in the new variables σ and γ. Using the chain rule, we can express the partial derivatives of $u(x,y)$ through the partial derivatives of $\widetilde{u}(\sigma,\gamma)$ as follows (subscripts stand for partial derivatives):

$$\begin{aligned}
u_t &= \widetilde{u}_\sigma\Sigma_t + \widetilde{u}_\gamma\Gamma_t = -c\widetilde{u}_\sigma + c\widetilde{u}_\gamma \\
u_{tt} &= \left(-c\widetilde{u}_\sigma + c\widetilde{u}_\gamma\right)_t = -c\left(\widetilde{u}_{\sigma\sigma}\Sigma_t + \widetilde{u}_{\sigma\gamma}\Gamma_t\right) + c\left(\widetilde{u}_{\gamma\sigma}\Sigma_t + \widetilde{u}_{\gamma\gamma}\Gamma_t\right) = c^2\left(\widetilde{u}_{\sigma\sigma} - 2\widetilde{u}_{\sigma\gamma} + \widetilde{u}_{\gamma\gamma}\right) \\
u_x &= \widetilde{u}_\sigma\Sigma_x + \widetilde{u}_\gamma\Gamma_x = \widetilde{u}_\sigma + \widetilde{u}_\gamma \\
u_{xx} &= \left(\widetilde{u}_\sigma + \widetilde{u}_\gamma\right)_x = \widetilde{u}_{\sigma\sigma}\Sigma_x + \widetilde{u}_{\sigma\gamma}\Gamma_x + \widetilde{u}_{\gamma\sigma}\Sigma_x + \widetilde{u}_{\gamma\gamma}\Gamma_x = \widetilde{u}_{\sigma\sigma} + 2\widetilde{u}_{\sigma\gamma} + \widetilde{u}_{\gamma\gamma}\ .
\end{aligned}$$

Plugging all these derivatives in the wave equation, we obtain

$$c^2\left(\widetilde{u}_{\sigma\sigma} - 2\widetilde{u}_{\sigma\gamma} + \widetilde{u}_{\gamma\gamma}\right) = c^2\left(\widetilde{u}_{\sigma\sigma} + 2\widetilde{u}_{\sigma\gamma} + \widetilde{u}_{\gamma\gamma}\right) \ ,$$

or, after elementary cancellations,

$$\widetilde{u}_{\sigma\gamma}(\sigma,\gamma) = 0 \ .$$

The only thing that you have to do in this part of the problem is to show that the general solution of this PDE is

$$\widetilde{u}(\sigma,\gamma) = f(\sigma) + g(\gamma) \ ,$$

where f and g are arbitrary functions of one variable.

(b) Go to the original variables to show that the general solution of the wave equation $u_{tt} = c^2\, u_{xx}$ is

$$u(x,t) = f(x-ct) + g(x+ct) \ .$$

(c) Now consider the initial value problem consisting of the wave equation in part (a) and the initial conditions

$$u(x,0) = \phi(x) \ , \qquad u_t(x,0) = \psi(x)$$

(where the subscript t stands for differentiation with the respect to t). In physical terms, $u(x,0)$ is the "initial position", and $u_t(x,0)$ is the "initial speed". Show that the functions f and g are related to ϕ and ψ as follows:

$$\begin{aligned} f(x) + g(x) &= \phi(x) \\ -f'(x) + g'(x) &= \frac{1}{c}\,\psi(x) \ . \end{aligned}$$

(d) One can integrate the second equation from the system in (c) and solve it for the functions f and g, obtaining

$$\begin{aligned} f(x) + g(x) &= \phi(x) \\ -f(x) + g(x) &= \frac{1}{c}\int_0^x \psi(s)\,ds + A \ , \end{aligned}$$

where A is an arbitrary constant. Solve this system to show that

$$\begin{aligned} f(x) &= \frac{1}{2}\phi(x) - \frac{1}{2c}\int_0^x \psi(s)\,ds - \frac{A}{2} \\ g(x) &= \frac{1}{2}\phi(x) - \frac{1}{2c}\int_0^x \psi(s)\,ds + \frac{A}{2} \ . \end{aligned}$$

(e) Use your result in (d) to show that the solution of the initial value problem

$$u_t t(x,t) = c^2\, u_{xx}(x,t) \ , \qquad x \in \mathbb{R} \ , \quad t \in [0,\infty)$$
$$u(x,0) = \phi(x) \ , \qquad u_t(x,0) = \psi(x) \ , \qquad \text{for } x \in \mathbb{R}$$

is

$$u(x,t) = \frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds \ .$$

Congratulations! You have derived the so-called *D'Alembert formula* for the solution of the wave equation in one spatial dimension on the whole real line!

(f) Use D'Alembert formula to solve the initial-value problem

$$u_t t(x,t) = c^2\, u_{xx}(x,t) \ , \qquad x \in \mathbb{R} \ , \quad t \in [0,\infty)$$
$$u(x,0) = 0 \ , \qquad u_t(x,0) = x\, e^{-x^2} \ , \qquad \text{for } x \in \mathbb{R} \ .$$

Problem 3. Solve the boundary value problem

$$\Delta u(x,y) = 0$$
$$u(0,y) = 0 \ , \qquad u(a,y) = 0 \qquad \text{for } y \in [0,\infty)$$
$$u(x,0) = \sin\frac{3\pi x}{a} \qquad \text{for } x \in [0,a] \ .$$

in the semi-infinite strip $x \in [0,a]$, $y \in [0,\infty)$. From physical point of view it is quite clear that we have to also impose the condition $\lim_{y\to\infty} u(x,y) = 0$.

Hint: When you are trying to find the functions $Y_n(y)$, it will be more convenient to write them as superposition of exponents rather than as superposition of hyperbolic functions (because $e^{-(\text{positive constant})\, y}$ tends to 0 while $e^{(\text{positive constant})\, y}$ tends to infinity as $y \to \infty$). *Reading Example 2 from Section 9.7 of the book (on pages 648, 649) will be VERY useful!*

Problem 4. Consider the following problem for the wave equation with air resistance term, with homogeneous Dirichlet BCs on the spatial interval $x \in [0,\pi]$:

$$u_{xx} - 10u_t - u_{tt} = 0 \ , \qquad x \in [0,\pi] \ , \quad t \geq 0 \ ,$$
$$u(0,t) = 0 \ , \quad u(\pi,t) = 0 \ , \qquad t \geq 0 \ ,$$
$$u(x,0) = -8\sin 3x + 12\sin 13x \ , \quad u_t(x,0) = 0 \ , \qquad x \in [0,\pi] \ .$$

Physically, this problem corresponds to a spring vibrating in air with resistance proportional to the velocity (i.e., to the time derivative $u_t(x,t)$). The coefficient multiplying $u_t(x,t)$ is proportional to the air resistance coefficient.

Because of the homogeneous Dirichlet BCs, it is clear that we should look for an expansion of the unknown function $u(x,t)$ of the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{L} = \sum_{n=1}^{\infty} T_n(t) \sin nx \tag{1}$$

(here $L = \pi$ is the length of the string).

(a) Plug the expansion (1) in the PDE to show that the unknown functions $T_n(t)$ must satisfy the ODEs

$$T_n''(t) + 10\, T_n'(t) + n^2 T_n(t) = 0 \ . \tag{2}$$

(b) The initial conditions for the functions $T_n(t)$ come from the initial conditions for $u(x,t)$. Plug the expansion (1) into the initial conditions for $u(x,t)$ to show that $T_n(0)$ and $T_n'(0)$ are zero for all n except for $n = 3$ and $n = 13$. What are the initial conditions $T_3(0)$ and $T_3'(0)$ for $T_3(t)$, and the initial conditions $T_{13}(0)$ and $T_{13}'(0)$ for $T_{13}(t)$?

(c) Since the ODEs (2) are homogeneous (i.e., have zero right-hand sides), the solutions for all functions $T_n(t)$ with n not equal to 3 or 13 will be identically equal to zero.

Solve the initial-value problem for the function $T_3(t)$.

(d) Solve the initial-value problem for the function $T_{13}(t)$.

(e) Write down the solution,

$$u(x,t) = T_3(t) \sin 3x + T_{13}(t) \sin 13x \ ,$$

with the functions $T_3(t)$ and $T_{13}(t)$ found in parts (c) and (d).

(f) From the physical interpretation of the problem, what would you expect the asymptotic position of the string to be. No calculation is needed here, only a couple of sentences of explanation.

(g) Does the solution found in part (e) behave as you predicted on physical grounds in part (f)?

Additional problem 1. (***<u>Not to be turned in; the solution is on the web-site!</u>***)
Consider the problem about the stationary temperature distribution in the rectangle $x \in [0,a]$, $y \in [0,b]$ (which can be symbolically written as $(x,y) \in [0,a] \times [0,b]$) if there are no sources of heat in the rectangle (hence the temperature $u(x,y)$ satisfies Laplace's equation $\Delta u = 0$), and the temperature at the sides of the rectangle is maintained as follows:

$$u(0,y) = 0 \ , \qquad u(a,y) = 0 \qquad \text{for } y \in [0,b]$$
$$u(x,0) = \sin \frac{3\pi x}{a} \ , \qquad u(x,b) = 5 \sin \frac{7\pi x}{a} \qquad \text{for } x \in [0,a] \ .$$

(a) Solve the boundary value problem

$$\begin{aligned}
&\Delta u = 0 \ , \qquad (x,y) \in [0,a]\times[0,b] \\
&u(0,y) = 0 \ , \qquad u(a,y) = 0 \qquad \text{for } y \in [0,b] \\
&u(x,0) = 0 \ , \qquad u(x,b) = 5\sin\frac{7\pi x}{a} \qquad \text{for } x \in [0,a] \ .
\end{aligned}$$

(b) Solve the boundary value problem

$$\begin{aligned}
&\Delta u = 0 \ , \qquad (x,y) \in [0,a]\times[0,b] \\
&u(0,y) = 0 \ , \qquad u(a,y) = 0 \qquad \text{for } y \in [0,b] \\
&u(x,0) = \sin\frac{3\pi x}{a} \ , \qquad u(x,b) = 0 \qquad \text{for } x \in [0,a] \ .
\end{aligned}$$

Hint: Let $Y_n(y)$ stands for the functions in the expansion

$$u(x,y) = \sum_{n=1}^{\infty} Y_n(y)\, X_n(x) \ ,$$

where because of the homogeneous boundary conditions at $x = 0$ and $x = a$ the functions $X_n(x)$ are given by $X_n(x) = \sin\frac{n\pi x}{a}$. Then the general solution of the ODE for $Y_n(y)$ is

$$Y_n(y) = C_n \cosh\frac{n\pi y}{a} + D_n \sinh\frac{n\pi y}{a} \ .$$

Show that the homogeneous boundary condition at $y = b$ implies that

$$\begin{aligned}
Y_n(y) &= E_n\left(\sinh\frac{n\pi b}{a}\cosh\frac{n\pi y}{a} - \cosh\frac{n\pi b}{a}\sinh\frac{n\pi y}{a}\right) \\
&= E_n \sinh\frac{n\pi(b-y)}{a}
\end{aligned}$$

(where E_n are constants arbitrary at the moment); here we have used the fact that hyperbolic sine satisfies

$$\sinh(\alpha \pm \beta) = \sinh\alpha\cosh\beta \pm \cosh\alpha\sinh\beta \ .$$

Now impose the remaining boundary condition to find the constants E_n (of which only one will be non-zero).

(c) Since the equation is linear and homogeneous (i.e., with a zero right-hand side), the principle of superposition holds similarly to the case of ordinary differential equations. Using this fact, write down the solution of the boundary value problem

$$\begin{aligned}
&\Delta u = 0 \ , \qquad (x,y) \in [0,a]\times[0,b] \\
&u(0,y) = 0 \ , \qquad u(a,y) = 0 \qquad \text{for } y \in [0,b] \\
&u(x,0) = \sin\frac{3\pi x}{a} \ , \qquad u(x,b) = 5\sin\frac{7\pi x}{a} \qquad \text{for } x \in [0,a] \ .
\end{aligned}$$