

# MATH 5453    Homework 10a    Not Due Thu, Nov 27

**Problems** 46, 48 from Section 2.5 of the book.

**Problems** 2, 3, 4, 5, 7 from Section 3.1 of the book.

**Additional problem 0.** Think about the proofs of all small remarks made in class (every time you heard the word “obviously” or “clearly”) about the concept of a signed measure, its positive, negative, and total variations, mutually singular signed measures, and a signed measure mutually singular with respect of a positive measure.

**Additional problem 1.** Let  $E = [1, \infty) \times (0, 1] \subset \mathbb{R} \times \mathbb{R}$ , and  $m$  stand for the Lebesgue measure on  $\mathbb{R}$ . Consider the function

$$f : E \rightarrow \mathbb{R} : (x, y) \mapsto f(x, y) = e^{-xy} - 2e^{-2xy} .$$

(a) Optional! Compute the integral

$$\int_{[1, \infty)} \left( \int_{(0, 1]} f(x, y) \, dm(y) \right) dm(x)$$

treating it as a Riemann integral. You can express the answer in terms of the function  $\Phi(z) := \int_1^z \frac{1}{t} e^{-zt} \, dt$ .

(b) Optional! Compute the integral

$$\int_{(0, 1]} \left( \int_{[1, \infty)} f(x, y) \, dm(x) \right) dm(y)$$

treating it as a Riemann integral. Again, express the answer in terms of the function  $\Phi$ , and use the fact that  $\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} \, dt = \ln \frac{b}{a}$  ( $a > 0, b > 0$ ).

(c) Explain why your answers in parts (a) and (b) were different, i.e., why the Fubini-Tonelli Theorem did not work.

**Additional problem 2.** Recall the following strategy of using Fubini-Tonelli Theorem together in order to reverse the order of integration in a double integral. Let  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$  be  $\sigma$ -finite measure spaces, and  $f : X \times Y \rightarrow \mathbb{C}$  be an  $\mathcal{M} \otimes \mathcal{N}$ -measurable function. In order to have

$$\int f \, d(\mu \times \nu) = \int \int f \, d\mu \, d\nu = \int \int f \, d\nu \, d\mu , \tag{1}$$

it will be enough (because of Fubini) to show that  $f \in L^1(\mu \times \nu)$ , i.e., that  $\int |f| \, d(\mu \times \nu) < \infty$ . Since  $f$  is  $\mathcal{M} \otimes \mathcal{N}$ -measurable,  $|f|$  is also  $\mathcal{M} \otimes \mathcal{N}$ -measurable, and also  $|f| \geq 0$ , so that  $|f| \in L^+(\mu \times \nu)$ . Thanks to Tonelli, we have

$$\int |f| \, d(\mu \times \nu) = \int \int |f| \, d\mu \, d\nu = \int \int |f| \, d\nu \, d\mu ,$$

so that if we show that either  $\int \int |f| d\mu d\nu$  or  $\int \int |f| d\nu d\mu$  is finite, this will imply that  $f \in L^1(\mu \times \nu)$ , and, hence, (1) will hold.

Consider the function

$$f : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R} : (x, y) \mapsto f(x, y) = \frac{e^{-x(y+1)} \sin(xy)}{\sqrt{y}}.$$

Apply the above strategy to show that for this function and the Lebesgue measure on  $\mathbb{R}^2$  the equation (1) holds.

*Remark:* The order in which you try to integrate in (1) matters, so do the things in the easiest possible way.