MATH 5763 Homework 10 Solved in class on 5/2/19

The *Ornstein-Uhlenbeck process* describes the velocity of a small particle in viscous fluid if the particle is subject to a rapidly fluctuating random force due to the very frequent collisions with the molecules of the fluid.

Let X_t stand for the velocity of the particle. The frictional force due to the viscosity of the fluid has magnitude proportional to the magnitude of the particle's velocity, and direction opposite to it, which gives the term $-k\,X_t$ in the right-hand side of equation (1) below. The random force due to the molecular collisions can be modeled by a generalized Gaussian white noise ξ_t which can be thought of as the (generalized) derivative of a standard Wiener process. Hence, the equation describing the evolution of X_t can be written in the form

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = -k\,X_t + \alpha\,\xi_t \ , \tag{1}$$

where $k > 0$ and $\alpha > 0$ are constants.

Assume that the initial velocity, X_0, of the particle is a random variable with known mean $\mathbb{E}[X_0]$, second moment $\mathbb{E}[X_0^2]$, and variance $\operatorname{var} X_0 = \mathbb{E}[X_0^2] - (\mathbb{E}[X_0])^2$.

(a) Write (1) in Itô form (i.e., as a stochastic differential equation); in other words, use that ξ_t is the generalized derivative of the Wiener process B_t, i.e., $\xi_t = \dfrac{\mathrm{d}B_t}{\mathrm{d}t}$. Put in the left-hand side all terms containing explicitly X_t, and multiply both sides of the equation by the integrating factor e^{kt}.

(b) Recall that if X_t satisfies the stochastic differential equation

$$\mathrm{d}X_t = f(t, X_t)\,\mathrm{d}t + g(t, X_t)\,dB_t \ , \tag{2}$$

then Itô formula reads

$$\mathrm{d}\Psi(t, X_t) = \left[\frac{\partial\Psi}{\partial t}(t, X_t) + \frac{1}{2}\,\frac{\partial^2\Psi}{\partial x^2}(t, X_t)\,g(t, X_t)^2\right]\mathrm{d}t + \frac{\partial\Psi}{\partial x}(t, X_t)\,\mathrm{d}X_t \ ,$$

or, equivalently (using (2)),

$$\begin{aligned}\mathrm{d}\Psi(t, X_t) \;&=\; \left[\frac{\partial\Psi}{\partial t}(t, X_t) + \frac{\partial\Psi}{\partial x}(t, X_t)\,f(t, X_t) + \frac{1}{2}\,\frac{\partial^2\Psi}{\partial x^2}(t, X_t)\,g(t, X_t)^2\right]\mathrm{d}t \\ &\quad + \frac{\partial\Psi}{\partial x}(t, X_t)\,g(t, X_t)\,\mathrm{d}B_t \ .\end{aligned}$$

Here the notations are the following:

$$\frac{\partial\Psi}{\partial x}(t, X_t) := \frac{\partial\Psi}{\partial x}(t, x)\Big|_{x=X_t} \ , \qquad \frac{\partial^2\Psi}{\partial x^2}(t, X_t) := \frac{\partial^2\Psi}{\partial x^2}(t, x)\Big|_{x=X_t} \ .$$

Use Itô formula to compute $\mathrm{d}(\mathrm{e}^{kt}X_t)$. Compare your result with the left-hand side of the equation you have obtained in (a), and show that the solution of (1) is

$$X_t = \mathrm{e}^{-kt}\left(X_0 + \alpha\int_0^t \mathrm{e}^{ks}\mathrm{d}B_s\right) \ . \tag{3}$$

(c) Directly from (3) find the expectation of X_t; use the fact that, for any function $h(t,x)$,

$$\mathbb{E}\left[\int_{t_0}^{t} h(t,X_s)\,\mathrm{d}B_s\right] = 0\ ,$$

and recall that you know the mean and the variance of the initial velocity X_0 of the particle.

(d) To find $\mathbb{E}[X_t^2]$ directly from the solution (3) of the stochastic differential equation (1), take expectation of the square of the right-hand side of (3), and use the properties of Itô integrals. Show that the variance of X_t is

$$\operatorname{var} X_t = \frac{\alpha^2}{2k} + \left(\operatorname{var} X_0 - \frac{\alpha^2}{2k}\right)\mathrm{e}^{-2kt}\ .$$

Hint: You will need to use the fact that X_0 is independent of the subsequent values of X_t, and the *Itô isometry*,

$$\mathbb{E}\left[\left(\int_{t_0}^{t} h(t,X_s)\,\mathrm{d}B_s\right)^2\right] = \mathbb{E}\left[\int_{t_0}^{t} h(t,X_s)^2\,\mathrm{d}s\right]\ .$$

(e) Here you will use a trickier method to find $\mathbb{E}[X_t^2]$ again. Namely, write Itô formula to find $\mathrm{d}(X_t^2)$, and express $\mathrm{d}X_t$ from the SDE you wrote in (a) in order to express $\mathrm{d}(X_t^2)$ in terms of $\mathrm{d}t$ and $\mathrm{d}B_t$. Then take expectation of both sides, and use the fact that $\mathbb{E}[\mathrm{d}(X_t^2)] = \mathrm{d}(\mathbb{E}[X_t^2])$ to show that the second moment, $n(t) := \mathbb{E}[X_t^2]$, of the stochastic process X_t satisfies the first-order linear ordinary differential equation

$$\frac{\mathrm{d}n(t)}{\mathrm{d}t} = \alpha^2 - 2k\,n(t)\ . \tag{4}$$

Solve this equation with the initial condition $n(0) = \mathbb{E}[X_0^2]$.
Hint: To solve (4), you may use the fact that e^{2kt} is an integrating factor for it.

(f) The Fokker-Planck equation for the conditional transition density function of the stochastic process X_t described by the stochastic differential equation (2) is

$$\frac{\partial p(x,x_0;t,t_0)}{\partial t} = -\frac{\partial}{\partial x}\left[f(t,x)\,p(x,x_0;t,t_0)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[g(t,x)^2\,p(x,x_0;t,t_0)\right]\ , \qquad t > t_0\ .$$

Write down the Fokker-Planck equation for the transition density for the process described by the Ornstein-Uhlenbeck equation (1). What initial condition should $p(x,x_0;t,t_0)$ satisfy?

(g) The Fokker-Planck equation obtained in part (f) is not easy to solve; its solution is

$$p(x,x_0;t,t_0) = \sqrt{\frac{k}{\pi\alpha^2}}\,\frac{1}{\sqrt{1-\mathrm{e}^{-2k(t-t_0)}}}\exp\left\{-\frac{k}{\alpha^2}\,\frac{\left(x-\mathrm{e}^{-k(t-t_0)}x_0\right)^2}{1-\mathrm{e}^{-2k(t-t_0)}}\right\} \tag{5}$$

One way to obtain useful information without actually solving the Fokker-Planck equation is to use it to derive an equation for the moment generating function of X_t conditioned on the event $\{X_{t_0} = x_0\}$,

$$M\big(\theta,t\big|X_{t_0}=x_0\big) := \mathbb{E}\big[\mathrm{e}^{\theta X_t}\big|X_{t_0}=x_0\big] = \int_{-\infty}^{\infty}\mathrm{e}^{\theta x}\,p(x,x_0;t,t_0)\,\mathrm{d}x$$

(here θ is just a dummy variable; before we used ξ to denote the dummy variable, but now ξ_t is used to denote the white noise).

Show that $M\big(\theta, t\big|X_{t_0} = x_0\big)$ satisfies the first order partial differential equation

$$\frac{\partial M}{\partial t} + k\theta\,\frac{\partial M}{\partial \theta} = \frac{\alpha^2}{2}\,\theta^2 M\ . \tag{6}$$

In the derivation of (6), all boundary terms that occur when you integrate by parts should be set equal to 0 (which corresponds to the fact that the conditional transition density p tends to 0 very fast as $|x| \to \infty$).

Show that the initial condition for $M\big(\theta, t\big|X_{t_0} = x_0\big)$ reads

$$M\big(\theta, t_0\big|X_{t_0} = x_0\big) = \mathrm{e}^{\theta x_0}\ .$$

(h) Solving first order partial differential equations is relatively straightforward (compared with solving second order PDEs like the Fokker-Planck equation). One can show that solution of (6) satisfying the initial condition you wrote in (g) is

$$M\big(\theta, t\big|X_{t_0} = x_0\big) = \exp\left\{x_0\mathrm{e}^{-k(t-t_0)}\,\theta + \frac{\alpha^2}{4k}\left(1 - \mathrm{e}^{-2k(t-t_0)}\right)\theta^2\right\}\ . \tag{7}$$

One can show that

$$\frac{\partial M}{\partial \theta}\big(0, t\big|X_{t_0} = x_0\big) = x_0\,\mathrm{e}^{-k(t-t_0)}\ , \qquad \frac{\partial^2 M}{\partial \theta^2}\big(0, t\big|X_{t_0} = x_0\big) = \frac{\alpha^2}{2k} + \left(x_0^2 - \frac{\alpha^2}{2k}\right)\mathrm{e}^{-2k(t-t_0)}\ .$$

Use these facts to find $\mathbb{E}[X_t|X_{t_0} = x_0]$ and $\operatorname{var}(X_t|X_{t_0} = x_0)$.

(i) One can use the expression for $M\big(\theta, t\big|X_{t_0} = x_0\big)$ found in part (h) to identify the distribution of the random variable X_t conditioned on the event $\{X_{t_0} = x_0\}$. To this end, recall that the moment generating function of a $\mathrm{N}(\mu, \sigma^2)$ random variable is

$$M_{\mathrm{N}(\mu,\sigma^2)}(\theta) = \mathrm{e}^{\mu\theta + \frac{1}{2}\sigma^2\theta^2}\ .$$

What is the type of this conditional distribution of X_t and what are its parameters?

(j) Look at your results from parts (h) and (i) and think about the physical interpretation of X_t. Explain whether the behavior of the expressions you obtained for $\mathbb{E}[X_t|X_{t_0} = x_0]$ and $\operatorname{var}(X_t|X_{t_0} = x_0)$ behave reasonably in each of the following limiting transitions:

$$t \downarrow t_0\ , \qquad t \to \infty\ , \qquad \alpha \to 0\ , \qquad k \to 0\ .$$

(k) Now assume that you did not know about the tricks in parts (g)–(i), but you were able to find the solution (5) of the Fokker-Planck equation for this problem. Assume also that you know the probability density function $f_{X_{t_0}}(x)$ of the random variable X_{t_0}. *Without doing any calculations*, tell me what equations you would use to compute the probability density function $f_{X_t}(x)$ of the random variable X_t in terms of these functions? Having found $f_{X_t}(x)$, how would you use it to compute $\mathbb{E}[X_t]$ and $\operatorname{var} X_t$? Again, just write the equations, without computing anything!