MATH 5763 Homework 10 Due Fri, 4/29/11

Problem 1. Let the random vector (X_1, X_2) has a binormal distribution with vector of means $\boldsymbol{\mu} = (\mu_1, \mu_2)$, variances σ_1^2 and σ_2^2, and correlation ρ, i.e., with covariance matrix

$$\mathbf{K} = \begin{pmatrix} \operatorname{Var} X_1 & \operatorname{Cov}(X_1, X_2) \\ \operatorname{Cov}(X_1, X_2) & \operatorname{Var} X_2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{pmatrix} .$$

Routine integrations show that $X_j \sim N(\mu_j, \sigma_j^2)$ $(j = 1, 2)$, the correlation between X_1 and X_2 is ρ, and X_1 and X_2 are independent if and only if they are uncorrelated (i.e., if $\rho = 0$).

(a) Show that the joint p.d.f. of X_1 and X_2 can be written as

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\} .$$

(b) Prove that $\mathbb{E}[X_1|X_2] = \mu_1 + \rho\,\frac{\sigma_1}{\sigma_2}\,(X_2 - \mu_2)$.

Hint: Concentrate on where x_1 occurs in $f_{X_1|X_2}(x_1|x_2)$; the overall constant can be sorted out later (if it is needed at all):

$$f_{X_1|X_2}(x_1|x_2) = C_1(x_2) \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1}{\sigma_1}\right)^2 - 2\frac{x_1}{\sigma_1}\frac{\mu_1}{\sigma_1} - 2\rho\frac{x_1}{\sigma_1}\frac{x_2-\mu_2}{\sigma_2}\right]\right\} ,$$

where $C_1(x_2)$ depends on x_2 only. Show that this expression can be written as

$$f_{X_1|X_2}(x_1|x_2) = C_2(x_2) \exp\left\{-\frac{[x_1-\mu_1-\rho\sigma_1(x_2-\mu_2)/\sigma_2]^2}{2(1-\rho^2)\sigma_1^2}\right\}$$

for some $C_2(x_2)$. This looks like the p.d.f. of a normally distributed random variable – what are the mean and the variance of this random variable?

(c) Use the hint to part (b) to show that that the variance of the conditional density function $f_{X_1|X_2}$ is equal to $\operatorname{Var}(X_1|X_2) = \sigma_1^2(1-\rho^2)$.

Problem 2. Let $X = \{X_t\}_{t\in\mathbb{R}}$ be a stationary, Gaussian, Markov process with zero means, and let $C_X(t)$ be the autocovariance function of the process (recall that for WSS processes $C_X(t)$ stands for $C_X(s, s+t)$ for any $s \in \mathbb{R}$). Use the results of Problem 1 to prove the properties below.

Remark: It is clear that a Gaussian process is SSS if and only if it is WSS. One can show that a Gaussian process is Markov if an only if

$$\mathbb{E}[X_{t_n}|X_{t_1} = x_1, \ldots, X_{t_{n-1}} = x_{n-1}] = \mathbb{E}[X_{t_n}|X_{t_{n-1}} = x_{n-1}] .$$

(a) Show that the result of Problem 1(b) yields $C_X(0)\,\mathbb{E}[X_{s+t}|X_s] = C_X(t)\,X_s \;\;\forall t \geq 0$.

(b) If $0 \le s \le s+t$, then fill in all the missing steps and explain what properties you use to derive the following chain of equalities:

$$\begin{aligned}
C_X(0)\,\mathbb{E}[X_0 X_{s+t}] &= C_X(0)\,\mathbb{E}\big[\,\mathbb{E}[X_0 X_{s+t}|X_0, X_s]\big] &(1)\\
&= C_X(0)\,\mathbb{E}\big[X_0\,\mathbb{E}[X_{s+t}|X_0, X_s]\big] &(2)\\
&= C_X(0)\,\mathbb{E}\big[X_0\,\mathbb{E}[X_{s+t}|X_s]\big] &(3)\\
&= C_X(t)\,\mathbb{E}[X_0 X_s]\ . &(4)
\end{aligned}$$

In particular, explain over what the outside averaging is on lines (1), (2), and (3). Where was the Markov property used?

(c) From (b) derive the identity $C_X(0)\,C_X(s+t) = C_X(s)\,C_X(t)$ for $s, t \ge 0$.

(d) Check that the function $C_X(t) = C_X(0)\,\mathrm{e}^{-\alpha|t|}$ satisfies the equation derived in (c) for any constant α (for "physical" reasons, it is clear that α should be positive). Briefly discuss this result in the light of Proposition 2.4.3 of Lefebvre's book.

Problem 3. Let B_t be a standard Wiener process, i.e., a process with independent increments and continuous sample paths satisfying the initial condition $B_0 = 0$, and such that $B_{s+t} - B_s$ is $N(0, (\sqrt{t})^2)$ for all $s \ge 0$ and $t > 0$. Define the process V_t by $V_0 = 0$, $V_t = tB_{1/t}$ for $t > 0$. Show that V_t is a standard Wiener process. It is enough to show that the process V_t has properties (a), (b), and (c) below.

(a) V_t is a continuous function of t with probability 1 (this is obvious from the definition of V_t for $t > 0$; the proof that $\lim_{t\downarrow 0} V_t = 0$ with probability 1 is a bit more complicated and you can skip it; at a "physical" level of rigor one has, for $t > 0$, $V_t = tB_t \sim t\,N(0, \frac{1}{t}) \sim N(0, t^2\frac{1}{t}) \sim N(0, t)$, which tends to 0 as $t \downarrow 0$).

(b) V_t has stationary increments (i.e., $V_b - V_a$ does not depend on s); moreover, for $0 \le a < b$, $V_b - V_a \sim N(0, b-a)$.
Hint: Write $V_b - V_a = bB_{1/b} - aB_{1/a} = (b-a)B_{1/b} - a(B_{1/a} - B_{1/b})$, then show that $(b-a)B_{1/b}$ and $a(B_{1/a} - B_{1/b})$ are independent normal random variables with zero mean and certain variances (find the variances), from which the desired property follows easily. Recall from elementary probability that, if $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ are constants and the random variables $X \sim N(\mu, \sigma^2)$ and $Y \sim N(\nu, \rho^2)$ are independent, then $\alpha X + \beta Y \sim N(\alpha\mu + \beta\nu, \alpha^2\sigma^2 + \beta^2\rho^2)$.

(c) V_t has independent increments, i.e., if $0 \le a < b < c < d$, then $V_b - V_a$ and $V_d - V_c$ are independent random variables. One way to show this is to prove that these random variables are uncorrelated, which will imply that they are independent (since they are normal). Since both $V_b - V_a$ and $V_d - V_c$ have zero mean, you have to show that $\mathbb{E}[(V_d - V_c)(V_b - V_a)] = 0$. Write $V_b - V_a$ as in part (b), and $V_d - V_c = (d-c)(B_{1/d} -$

$B_0) - c(B_{1/c} - B_{1/d})$, and then use the properties of the standard Wiener process to show that

$$\begin{aligned}
\mathbb{E}[(V_b - V_a)(V_d - V_c)] &= (d-c)(b-a)\,\mathbb{E}[B_{1/d}B_{1/b}] - c(b-a)\,\mathbb{E}[(B_{1/c} - B_{1/d})B_{1/b}] \\
&= (d-c)(b-a)\frac{1}{d} - c(b-a)\left(\frac{1}{c} - \frac{1}{d}\right) = 0\ .
\end{aligned}$$

Please fill in the missing steps of the calculations and explain what properties you use at each step.

Remark: One can also prove that $B_{s+t} - B_s$ (for any fixed $s \geq 0$) and $\alpha B_{t/\alpha^2}$ (for any positive constant α) are standard Wiener processes. (You don't need to do this here.)

Problem 4. Recall that for a standard Wiener process B_t, the random vector $(B_{t_1}, \ldots, B_{t_n})$ has a multivariate normal distribution for any $0 \leq t_1 < t_2 < \cdots < t_n$. Also, by definition, the increments of the Wiener process over non-overlapping time intervals are independent, and $B_t - B_s$ is an $N\left(0, (\sqrt{t-s})^2\right)$ random variable for $0 \leq s < t$; as usual, $B_0 = 0$.
Let

$$f(x_n, x_{n-1}, \ldots, x_1;\, t_n, t_{n-1}, \ldots, t_1) = f_{(B_{t_n}, B_{t_{n-1}}, \ldots, B_{t_1})}(x_n, x_{n-1}, \ldots, x_1)$$

stand for the joint densities of the Wiener process at times $0 \leq t_1 < t_2 < \cdots < t_n$), where $f_{(B_{t_n}, B_{t_{n-1}}, \ldots, B_{t_1})}$ is the joint probability density of the random vector $(B_{t_n}, B_{t_{n-1}}, \ldots, B_{t_1})$ (although the letter f is terribly overloaded, its meaning is always clear from the context and from the arguments of the function). Let

$$p(x, x_0; t, t_0) = \frac{f(x, x_0; t, t_0)}{f(x_0, t_0)} = f_{B_t|B_{t_0}}(x|x_0) \qquad \text{for} \quad t_0 < t$$

be the conditional transition density function for the process B_t. Let $0 < s < t$. In this problem you will find the conditional density of the standard Wiener process at time t given its value at the earlier time t_0 (this is very easy), and the conditional density of the Wiener process at time t_0 given its value at the later time t (this is a little bit more complicated).

(a) If we know that $B_{t_0} = x_0$, then what is the distribution of B_t at the later moment $t > t_0$? Write explicitly the density of B_t conditioned on the event $\{B_{t_0} = x_0\}$, and identify its type and parameter values. Explain briefly your reasoning.

Hint: What type of random variable is $B_t - B_{t_0}$?

(b) Write explicitly the probability density, $f(x_0, t_0) = f_{B_{t_0}}(x_0)$, of B_{t_0}. Explain briefly.

Hint: What type of random variable is B_{t_0}?

(c) Write explicitly the joint probability density $f(x, x_0;\, t, t_0) = f_{(B_t, B_{t_0})}(x, x_0)$. Explain briefly.

(d) If you know that at the later time t the Wiener process has value $B_t = x$, find the conditional probability density $f_{W_{t_0}|W_t}(x_0|x)$ of the process at the earlier moment t_0. Can you recognize the type and the parameter of the random variable B_{t_0} given that $B_t = x$?

Hint: You may need to use the identity

$$\frac{x_0^2}{t_0} + \frac{x^2}{t - t_0} - \frac{2xx_0}{t - t_0} + \frac{x_0^2}{t - t_0} - \frac{x^2}{t} = \frac{\left(x_0 - \frac{t_0}{t}x\right)^2}{\frac{t_0}{t}(t - t_0)} .$$

Problem 5. Use the definition of the Dirac δ-function and its derivatives to compute the following integrals (where we used the standard notation $\delta := \delta_0$):

$$\int_{\mathbb{R}} x^7\,\delta(x)\,\mathrm{d}x\ , \qquad \int_{\mathbb{R}} \mathrm{e}^{3x^4}\,\delta(x)\,\mathrm{d}x\ , \qquad \int_{\mathbb{R}} \mathrm{e}^{3x^4}\,\delta'(x)\,\mathrm{d}x\ , \qquad \int_{\mathbb{R}} \mathrm{e}^{3x^4}\,\delta'(x-5)\,\mathrm{d}x\ .$$