MATH 5463 Homework 10 Due Tue, May 4

Problem 1. Let $\psi : \mathbb{R} \to \mathbb{R}$ be the $C_c^\infty(\mathbb{R})$ function with $\operatorname{supp}\psi = [-1, 1]$ (this function was constructed on page 236 of Folland, but you don't need its explicit form, just its properties).

(a) Define the sequence $\{f_k\}_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ by $f_k(x) = \dfrac{1}{k}\cos\left(\dfrac{k\pi x}{2}\right)\psi(x)$. Explain why sequence does *not* converge to 0 in $\mathcal{D}(\mathbb{R})$, although pointwise $f_k \to 0$.

(b) Let $\{g_k\}_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ be a sequence defined by $g_k(x) = \mathrm{e}^{-k}\,\psi(x-k)$. Does this sequence converge in the uniform metric? Does it converge in $\mathcal{D}(\mathbb{R})$? Explain briefly.

Problem 2. For $t > 0$ define the functions

$$f_t : \mathbb{R} \to \mathbb{R} : x \mapsto f_t(x) = \frac{1}{t}\,\chi_{[0,t]}(x) \ .$$

Clearly, each f_t is in $L^1_{\mathrm{loc}}(\mathbb{R})$, and, thus, defines a distribution $f_t \in \mathcal{D}'(\mathbb{R})$ by

$$\langle f_t, \phi\rangle = \int_{\mathbb{R}} f_t(x)\,\phi(x)\,\mathrm{d}x \ , \qquad \phi \in \mathcal{D}(\mathbb{R}) \ .$$

(a) Find $\lim\limits_{t\to 0+}\langle f_t, \phi\rangle$. Can you write your result as $\int \phi\,\mathrm{d}\mu$ for some measure μ?

(b) If τ_s stands for the translation operator (see page 238 of Folland), find $\langle \tau_s f_t, \phi\rangle$ and the limits $\lim\limits_{t\to 0+}\langle \tau_s f_t, \phi\rangle$ and $\lim\limits_{s\to 0}\lim\limits_{t\to 0+}\left\langle \dfrac{\tau_s f_t - f_t}{s}, \phi\right\rangle$ Can you write your results in the form $\int \phi\,\mathrm{d}\mu$ for some measures μ?

Problem 3. Let $X = [0, 1]$ and $\mathcal{M}$ be the σ-algebra of Borel subsets of X. Let $F(t) = t^2$, $G(t) = t^3$, and define the measures ϕ and γ on $\mathcal{M}$ by

$$\phi(E) = \int_E 1\,\mathrm{d}\mu_F \ , \quad \gamma(E) = \int_E 1\,\mathrm{d}\mu_G \ , \qquad E \in \mathcal{M} \ .$$

Does $\dfrac{\mathrm{d}\phi}{\mathrm{d}\gamma}$ exist? Does $\dfrac{\mathrm{d}\gamma}{\mathrm{d}\phi}$ exist? Compute the values of the Radon-Nikodym derivatives that exist. Justify.

Problem 4. Consider the *heat equation* in one time and one space dimension,

$$\partial_t u(t,x) = \partial_{xx} u(t,x) \ , \qquad t > 0 \ , \quad x \in \mathbb{R}$$

(the name comes from the fact that it describes the propagation of heat in a homogeneous medium without heat sources in one spatial dimension; here $u(t,x)$ is the temperature of the medium at time t at the location with spatial coordinate x). Usually one imposes an initial condition, i.e., a function u_0 such that $\lim\limits_{t\to 0+} u(t,x) = u_0(x)$, and sometimes also conditions on the behavior of the solution as $x \to \pm\infty$.

(a) Let for $t > 0$ and $x \in \mathbb{R}$, the function $u(t, \cdot)$ be a solution of the heat equation at time t that tends to zero fast enough as $|x| \to \infty$. Let $\widehat{u}(t, \cdot)$ be the Fourier transform of u with respect to the spatial variable: $\widehat{u}(t, \xi) = \displaystyle\int_{\mathbb{R}} u(t, x)\, \mathrm{e}^{-2\pi i \xi x}\, \mathrm{d}x$; the inverse Fourier transform is $u(t, x) = \displaystyle\int_{\mathbb{R}} \widehat{u}(t, \xi)\, \mathrm{e}^{2\pi i \xi x}\, \mathrm{d}\xi$ (assuming that the functions are in appropriate function spaces, so that the Fourier transform and its inverse are well defined; e.g., if $u(t, \cdot) \in \mathcal{S}$ and $\widehat{u}(t, \cdot) \in \mathcal{S}$).

Plug $u(t, \cdot)$ in the heat equation to show that $\widehat{G}_t$ satisfies the ordinary differential equation $\dfrac{\mathrm{d}\widehat{u}(t, \xi)}{\mathrm{d}t} = -4\pi^2\xi^2$, and find the general solution of this equation; this solution will contain one arbitrary function of ξ, let's denote it by $A(\xi)$.

(b) Now plug u in the initial condition $\lim_{t \to 0+} u(t, x) = u_0(x)$, and find the function $A(\xi)$. Write the solution of the initial value problem consisting of the heat equation and the initial condition.

(c) In the rest of this problem you will obtain an important general form of the solutions of the heat equation. Let $G_t(x)$ stand for the solution of the heat equation corresponding to the choice $A(\xi) \equiv 1$. If $\widehat{G}_t(x) = \int_{\mathbb{R}} G_t(x)\, \mathrm{e}^{-2\pi i \xi x}\, \mathrm{d}x$ is the Fourier transform of G_t) (in the spatial variable), then what is $\widehat{G}_t$? (This is a trivial question.)

Use the fact that, if $f : \mathbb{R} \to \mathbb{R}$ is the function given by $f(x) = \mathrm{e}^{-\pi a x^2}$ (where $a > 0$ is a constant), then $\widehat{f}(\xi) = \frac{1}{\sqrt{a}} \mathrm{e}^{-\pi \xi^2 / a}$ (this is Proposition 8.24 in Folland for $n = 1$), to show that $G_t(x) = \dfrac{1}{\sqrt{4\pi t}}\, \mathrm{e}^{-\frac{x^2}{4t}}$.

(d) Check by direct substitution that the function G_t obtained in (c) is a solution of the heat equation.

(e) An *approximate identity* in $\mathbb{R}$ is a family of functions $\{K_t\}_{t>0}$, where $K_t : \mathbb{R} \to \mathbb{R}$, with the following properties:

$$\sup_t \int_{\mathbb{R}} |K_t(y)|\, \mathrm{d}y < \infty \ , \qquad \lim_{t \to 0+} \int_{\mathbb{R}} K_t(y)\, \mathrm{d}y = 1 \ , \qquad \lim_{t \to 0+} \int_{\{|y| > \delta\}} |K_t(y)|\, \mathrm{d}y = 0 \ ,$$

for any constant $\delta > 0$. (This definition of an approximate identity looks different from the one in Folland, but in fact they are not so different.)

Check that $\{G_t\}_{t>0}$ is an approximate identity. You may use that $\int_{\mathbb{R}} \mathrm{e}^{-x^2}\, \mathrm{d}x = \sqrt{\pi}$.

(f) Prove that the function $u(t, x) := (G_t * u_0)(x)$ satisfies the heat equation and the boundary condition $\lim_{t \to 0+} u(t, x) = u_0(x)$, for any u_0 that decays fast enough at infinity. Does one need to impose differentiability conditions on u_0? Explain briefly.