MATH 4163 Homework 10 Due Tue, 04/17/2018

Problem 1. [Laplace's equation in a rectangle with Dirichlet/Neumann BCs]
Solve the following BVP for the Laplace's equation:

$$\begin{aligned} &\Delta u(x,y) = 0 \ , \qquad x \in [0,a] \ , \quad y \in [0,b] \ , \\ &u_x(0,y) = 0 \ , \quad u_x(a,y) = 0 \ , \\ &u(x,0) = 0 \ , \quad u(x,b) = f(x) \ , \end{aligned} \tag{1}$$

where the function f has Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{a} \ , \qquad a_j = \frac{2}{a}\int_0^a f(x)\,\cos\frac{j\pi x}{a}\,\mathrm{d}x \ , \quad j \in \{0,1,2,\ldots\} \ .$$

The physical meaning of the BVP (1) is the following: $u(x,y)$ can be interpreted as the steady-state temperature distribution in the rectangle $[0,a]\times[0,b]$ such that:

- there are no heat sources inside the rectangle $[0,a]\times[0,b]$;
- the walls at $x=0$ and $x=a$ are thermally insulated (i.e., the heat flux through each point of these walls is zero);
- the temperatures at the wall at $y=0$ is kept equal to zero, while the temperature at the wall at $y=b$ is given by the function $f(x)$.

You may use that the BVP

$$\begin{aligned} &X''(x) - \mu X(x) = 0 \ , \qquad x \in [0,a] \ , \\ &X'(0) = 0 \ , \quad X'(a) = 0 \end{aligned}$$

has a non-zero solution only if μ takes one of the values

$$\mu_n = -\left(\frac{n\pi}{a}\right)^2 \ , \qquad n \in \{0,1,2,\ldots\} \ .$$

The corresponding solutions of this BVP are

$$X_n(x) = \begin{cases} 1 & \text{if } n = 0 \ , \\ \cos\dfrac{n\pi x}{a} & \text{if } n \in \mathbb{N} \ . \end{cases} \tag{2}$$

When you separate variables, the solution $u(x,y)$ of the BVP (1) is a superposition of functions of the form

$$u_n(x,y) = X_n(x)\,Y_n(y) \ , \tag{3}$$

where the functions X_n are given by (2).

(a) What ODEs do the functions Y_n in (3) satisfy? (Derive the ODEs separately for the cases $n = 0$ and $n \in \mathbb{N}$.)

The general solutions of the ODEs you just wrote are

$$Y_n(y) = \begin{cases} A_0 + B_0 y & \text{if } n = 0 \ , \\ A_n \cosh \dfrac{n\pi y}{a} + B_n \sinh \dfrac{n\pi y}{a} & \text{if } n \in \mathbb{N} \end{cases} \tag{4}$$

(*you do not need to derive this!*). We could have written the solution for $Y_n(y)$ for $n \in \mathbb{N}$ as a sum of two exponents, but the representation in (4) is more convenient in this problem.

(b) Do the functions $u_n(x, y) = X_n(x)\, Y_n(y)$ satisfy the PDE from the BVP (1)? Why? Which of the four BCs does each of these functions satisfy?

(c) Write the expansion

$$u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) = \sum_{n=0}^{\infty} X_n(x)\, Y_n(y) \ ,$$

with the explicit expressions for X_n and Y_n. Impose the remaining BCs from the BVP (1) to find the constants in the functions Y_n. Write down the solution $u(x, y)$ of the BVP (1).

(d) Solve the BVP (1) in the case

$$f(x) = 5 + 3\cos\frac{7\pi x}{a} \ .$$

Problem 2. [Laplace's equation in a rectangle with Neumann BCs on all walls]
In this problem you will attempt to solve the following BVP for the Laplace's equation:

$$\begin{aligned} &\Delta u(x, y) = 0 \ , \qquad x \in [0, a] \ , \quad y \in [0, b] \ , \\ &u_x(0, y) = 0 \ , \quad u_x(a, y) = 0 \ , \\ &u_y(x, 0) = 0 \ , \quad u_y(x, b) = f(x) \ , \end{aligned} \tag{5}$$

where the function f has Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{a} \ , \qquad a_j = \frac{2}{a}\int_0^a f(x)\, \cos\frac{j\pi x}{a}\, \mathrm{d}x \ , \quad j \in \{0, 1, 2, \ldots\} \ . \tag{6}$$

The BVP (5) is similar to the BVP (1) considered in Problem 1, but here the BCs on *all* walls are Neumann BCs (i.e., the derivative of $u(x, y)$ in a direction normal to the wall is given), while in Problem 1 the BCs on two walls were Neumann and on the other two walls the BCs were of Dirichlet type (i.e., the value of $u(x, y)$ at the wall is given). This difference seems very small, but in fact is crucial because of the physical interpretation of the BVPs (1) and (5).

The solution of this problem is very similar to the one of Problem 1 (some things are totally identical), so use your results from Problem 1 without rederiving them here. Assume that, again, we look for the solution $u(x, y)$ as a superposition of functions $u_n(x, y) = X_n(x)\, Y_n(y)$. The functions X_n are again given by (2). For the rest of the problem, follow the steps below.

(a) What ODEs do the functions Y_n satisfy? What are the general solutions of these ODEs?

Hint: How is this part of the problem different from part (a) of Problem 1?

(b) Do the functions $u_n(x, y) = X_n(x)\, Y_n(y)$ satisfy the PDE from the BVP (5)? Which of the four BCs does each of these functions satisfy?

(c) As in Problem 1(c), write the expansion

$$u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) = A_0 + B_0 y + \sum_{n=1}^{\infty} \left(A_n \cosh \frac{n\pi y}{a} + B_n \sinh \frac{n\pi y}{a} \right) \cos \frac{n\pi x}{a}$$

and impose the remaining BCs in (5) to derive equations for the constants A_j and B_j. Do *not* solve the equations here!

(d) Show that the BC at $y = 0$ imply that $B_j = 0$ for all $j = 0, 1, 2, \ldots$.

Hint: If

$$c_0 + \sum_{n=1}^{\infty} c_n \cos \frac{n\pi x}{a} = 0 \qquad \text{for all } x \in [0, a] \;, \tag{7}$$

then you can conclude that $c_j = 0$ for all $j = 0, 1, 2, \ldots$. This can be derived simply (*but you do not need to do this!*): using the fact that the system of functions $\{1, \cos \frac{\pi x}{a}, \cos \frac{2\pi x}{a}, \cos \frac{3\pi x}{a}, \ldots\}$ is orthogonal on $[0, a]$ with respect to the inner product

$$\langle f, g \rangle = \int_0^a f(x)\, g(x)\, \mathrm{d}x \;,$$

you can multiply (7) consecutively by 1, $\cos \frac{\pi x}{a}$, $\cos \frac{2\pi x}{a}$, ..., to show that all coefficients c_j must be 0.

(e) Now that you know that all $B_j = 0$ for $j = 0, 1, 2, \ldots$, impose the BC at $y = b$ to try to find the coefficients A_j. Equate the expression for $u_y(x, b)$ (with $B_j = 0$) to the function $f(x)$ from (6). What do you get for the coefficients A_n for $n \in \mathbb{N}$?

(f) The most interesting thing here is what you obtain for the coefficient A_0. Do you obtain any condition for it? What would happen if the coefficient a_0 in (6) is not equal to zero?

Hint: The answer to the last question is very dramatic!

(g) The physical reason for your dramatic answer in part (f) is that the function $f(x)$ in (5) gives the flux of heat energy through the wall at $y = b$. The coefficient a_0 of the Fourier cosine series (6) of $f(x)$ is proportional to the average of the function $f(x)$ over the interval $x \in [0, a]$: indeed,

$$\frac{1}{a}\int_0^a f(x)\,\mathrm{d}x = \frac{1}{a}\int_0^a \left(\frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{a}\right)\mathrm{d}x = \frac{a_0}{2} \ .$$

If $a_0 \neq 0$, this means that the net amount of heat going into the rectangle $[0, a] \times [0, b]$ through the wall at $y = b$ is non-zero, while the other three walls are thermally insulated. Recall that Laplace's equation describes the steady-state heat distribution. What is the physical explanation of the fact that if $a_0 \neq 0$, the BVP (5) has no solution?

Problem 3. [Laplace's equation in an annulus]

Use the expression derived in class for the solution of Laplace's equation in an annulus (also look at Proposition 2 on page 370 of the book by Bleecker and Csordas) to find the solution of the boundary value problem

$$\begin{aligned}
&\Delta u(r,\theta) = 0 \ , \qquad r \in [1,2] \ , \quad \theta \in [0, 2\pi) \ , \\
&u(1,\theta) = 3 \ , \\
&u(2,\theta) = 7 + 5\sin 3\theta \ .
\end{aligned} \tag{8}$$

In this problem you may simply use the representation of the solution as an infinite series without writing the derivation, and only match the coefficients. However, I do expect you to go through your lecture notes and the text from the book and identify the main points in the derivation of this expression. For example, note that in this problem the discretization of the constant occurring in the separation of variables comes not from a boundary condition, but from the fact that the angular part $\Theta(\theta)$ of the solution $u(r,\theta) = R(r)\Theta(\theta)$ must be a periodic function of period 2π.

Problem 4. [Shape of a hanging circular membrane]

Consider a circular membrane of radius a attached at the rim, hanging in the gravity field. If the membrane is at rest, its shape can be described by the equation $z = u(r, \theta)$, where z is the vertical coordinate, and (r, θ) are polar coordinates in the (x, y)-plane. The function u satisfies Poisson's equation

$$\begin{aligned}
&\tau\Delta u(r,\theta) - \rho g = 0 \ , \qquad x \in [0, L] \ , \\
&u(a,\theta) = 0 \ ,
\end{aligned} \tag{9}$$

where τ is a quantity related to the tension in the membrane (unit: N/m), ρ is the mass density (mass per unit area) of the membrane (unit: kg/m^2), and g is the free-fall acceleration.

Since the membrane is circular and the gravity pull is uniform, it is clear that the shape of the membrane will not depend on the angular coordinate θ, so that we can set $u(r,\theta) = R(r)$.

(a) Using that the Laplacian in polar coordinates is given by

$$\Delta u(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\,\frac{\partial^2 u}{\partial \theta^2}\ ,$$

write the ODE and the BC that the function $R(r)$ and must satisfy.

(b) Find the general solution of the ODE written in part (a); it should contain two arbitrary constants.

(c) In the general solution of the ODE for $R(r)$ there is a term that is unbounded when $r \to 0^+$. This term should not be there. Write down the terms in $R(r)$ that behave in a physically reasonable way.

(d) Impose the boundary condition on $R(r)$ to find $R(r)$.

(e) Find the maximum hanging of the membrane, i.e., $\max\limits_{r\in[0,a]} |u(r,\theta)|$.

Problem 5. [Rotationally symmetric waves on a circular membrane]
Consider the IBVP describing the rotationally symmetric – i.e., independent of θ – waves in a circular membrane of radius a firmly attached at the rim. For such solution to exist, we have to assume that the initial conditions do not depend on θ, so that the problem has the form

$$\begin{aligned} &u_{tt}(r,t) = c^2\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right)\ , \qquad r\in[0,a]\ ,\quad t\in\mathbb{R}_+\ ,\\ &u(a,t)=0\ ,\quad |u(r,t)| \text{ is bounded}\ ,\\ &u(r,0)=f(r)\ ,\quad u_t(r,0)=g(r)\ . \end{aligned} \tag{10}$$

(a) We separate variables in the PDE (10) as usual: set $u(r,t) = R(r)\,T(t)$. The sign of the separation of variables constant must be such that the function $T(t)$ must be oscillatory, i.e., $T(t)$ must satisfy the ODE $T''(t)+c^2\lambda^2 T(t) = 0$ whose general solution is $T(t) = C_1\cos(c\lambda t)+C_2\sin(c\lambda t)$. Here λ is a constant which can be assumed positive without loss of generality: $\lambda > 0$. As usual, the constant λ can take only a discrete set of values that will be found later. What ODE does the function $R(r)$ satisfy?

(b) Recall from Problem 5 of Homework 3 and Problem 4 of Homework 9 that the ODE

$$w''(z) + \tfrac{1}{z}\,w'(z) + \left(1-\tfrac{n^2}{z^2}\right)w(z) = 0, \quad n = 0,1,2,3,\ldots\ , \tag{11}$$

is called the *Bessel differential equation*. The general solution of (11) is usually written as

$$w(z) = A\,J_n(z) + B\,Y_n(z)\ ,$$

where the functions $J_n(z)$ and $Y_n(z)$ called respectively *Bessel functions* and *Neumann functions* (or Bessel functions of first, resp., second kind); see Homework 9 for the graphs of $J_n(z)$ and $Y_n(z)$. The important facts to notice are: $J_0(0) = 1$, $J_0'(0) = 0$, $J_n(0) = 0$ for $n = 1, 2, 3, \ldots$, while all functions $Y_n(z)$ tend to $-\infty$ as $z \to 0^+$.

Change the variable r to $z := \lambda r$, and the unknown function $R(r)$ to $Z(z) := R\left(\frac{z}{\lambda}\right)$ to show that the function $Z(z)$ satisfy the Bessel ODE with $n = 0$, so that $Z(z)$ is a linear combination of the functions $J_0(z)$ and $Y_0(z)$. Then change the independent variable back to r to find $R(r)$.

(c) Some of the functions obtained in part (b) behave non-physically. Which ones? Write down the expression for $R(r)$ if we want it to behave in a physically reasonable way.

(d) Impose the BC at $r = a$, and obtain the values that λ can take. Let ξ_{0k} ($k = 1, 2, 3, \ldots$) be the kth zero of $J_0(\xi)$, i.e., $J_0(\xi_{0k}) = 0$, ordered so that $\xi_{01} < \xi_{02} < \cdots$.

(e) Write the functions $T_k(t)$. What are the frequencies of the functions $T_k(t)$? Recall that the frequency ν of a periodic function $\phi(t)$ of time with period P (i.e., satisfying $\phi(t) = \phi(t + P)$ for every $t \in \mathbb{R}$) is the inverse of the period: $\nu = \dfrac{1}{T}$.

(f) Write the function $u(r, t)$ giving the shape of the vibrating membrane. (We have not imposed initial conditions, so that your expression will contain arbitrary constants.)

Problem 6. [A Sturm-Liouville problem with Neumann/Robin BCs]
Consider the Sturm-Liouville problem

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}\left(x\,\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \frac{\lambda}{x}\,y(x) = 0\ , \qquad x \in [1, L]\ , \\ &y'(1) = 0\ , \qquad y'(L) + \beta y(L) = 0\ , \end{aligned} \tag{12}$$

where $L > 1$ and $\beta > 0$ are constants. Before solving the problem below, it will be useful to reread the Appendix to Homework 6.

(a) Let $y_n(x)$, $n = 1, 2, 3, \ldots$ be the eigenfunctions of the eigenvalue problem (12). Write down the orthogonality relation that the general theory guarantees that the functions $y_n(x)$ satisfy.

(b) Prove that the change of the independent variable

$$x = \mathrm{e}^{\widetilde{x}}\ , \quad \text{or, equivalently,} \quad \widetilde{x} = \ln x\ ,$$

and the corresponding change of the function

$$\widetilde{y}(\widetilde{x}) = y\left(\mathrm{e}^{\widetilde{x}}\right)\ , \quad \text{or, equivalently,} \quad y(x) = \widetilde{y}\left(\ln x\right)\ ,$$

transforms the ODE from (12) into

$$\frac{\mathrm{d}^2\widetilde{y}}{\mathrm{d}\widetilde{x}^2} + \lambda\widetilde{y} = 0 \; . \tag{13}$$

Please write your calculations in detail.

(c) Carefully write down the boundary conditions from (12) in terms of the function $\widetilde{y}\,(\widetilde{x})$; note that $\widetilde{x} \in [0, \ln L]$.

(d) We have seen many times that the ODE (13) for $\widetilde{y}\,(\widetilde{x})$ with homogeneous boundary conditions must have as solutions solutions trigonometric functions, not exponents, so that it must have the form

$$\frac{\mathrm{d}^2\widetilde{y}}{\mathrm{d}\widetilde{x}^2} + \alpha^2\widetilde{y} = 0 \tag{14}$$

for some constant α which can be assumed nonnegative without loss of generality. The general solution of (14) is

$$\widetilde{y}\,(\widetilde{x}) = A\cos\left(\alpha\widetilde{x}\right) + B\sin\left(\alpha\widetilde{x}\right)$$

for some constants A and B.

(e) Use the boundary conditions for $\widetilde{y}\,(\widetilde{x})$ derived in part (c) to find the allowed values of the constants α from (14). You will find that they satisfy a nonlinear equation involving the function tan.

(f) Draw a picture to illustrate that the nonlinear equation for α has infinitely many positive roots (recall Problem 3(b) from Homework 4).

(g) Let α_n $(n = 1, 2, 3, \ldots)$ be the roots of the equation for α derived in part (e). Go back to the original function $y(x)$ and write down the eigenfunction $y_n(x)$ and the eigenvalue λ_n corresponding to the value α_n.