MATH 4073 Homework 10 Due $\boxed{\text{in class}}$ on Fri, 12/09/16

Problem 1. Let the function $y(t)$ be the solution of the initial value problem

$$
\begin{aligned}
&\frac{\mathrm{d}y}{\mathrm{d}t}(t) = f(t, y(t)) \ , \qquad t \in [a, b] \ , \\
&y(a) = \alpha \ .
\end{aligned} \tag{1}
$$

Using that the function $y(t)$ satisfies (1), we obtain that the Taylor expansion of $y(t+h)$ has the form

$$
\begin{aligned}
y(t+h) &= y(t) + \frac{1}{1!} y'(t)\, h + \frac{1}{2!} y''(t) h^2 + O(h^3) \\
&= y(t) + f(t, y(t))\, h + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} f(t, y(t))\, h^2 + O(h^3) \\
&= y(t) + f(t, y(t))\, h + \frac{1}{2} \left[\frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t))\, y'(t) \right] h^2 + O(h^3) \\
&= y(t) + f(t, y(t)) h \ + \frac{1}{2} \left[\frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t))\, f(t, y(t)) \right] h^2 + O(h^3) \ .
\end{aligned}
$$

This gives rise to the Taylor method of order 2 that has the form

$$
\begin{aligned}
w_0 &= \alpha \\
w_{i+1} &= w_i + f(t_i, w_i) h + \frac{1}{2} \left[\frac{\partial f}{\partial t}(t_i, w_i) + \frac{\partial f}{\partial y}(t_i, w_i)\, f(t_i, w_i) \right] h^2 \ , \qquad i = 0, 1, \ldots, n-1 \ ;
\end{aligned}
$$

one can show that this method has local truncation error $O(h^2)$.

(a) Assume that the function $y(t)$ is a solution of the initial value problem (1), and obtain the Taylor expansion of $y(t+h)$ of the form

$$
\begin{aligned}
y(t+h) &= y(t) + \frac{1}{1!} y'(t) h + \frac{1}{2!} y''(t) h^2 + \frac{1}{3!} y'''(t) h^3 + O(h^4) \\
&= y(t) + [\text{something}]\, h + [\text{something}]\, h^2 + [\text{something}]\, h^3 + O(h^4) \ ,
\end{aligned}
$$

where each "something" stands for an expression containing the function f and its partial derivatives (up to total order 2), but *not* derivatives of the function $y(t)$. Do the necessary calculations and replace each "something" with the corresponding expression. I want to see your detailed calculations. Please write down the functions with all their arguments.

(b) Use your result from part (a) to write down a Taylor method of order 3 (with local truncation error $O(h^3)$) in the form

$$
\begin{aligned}
w_0 &= \alpha \\
w_{i+1} &= w_i + [\text{something}]\, h + [\text{something}]\, h^2 + [\text{something}]\, h^3 \ .
\end{aligned}
$$

Please be specific.

Problem 2. Consider the IVP

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= y^2 + \frac{1}{t^2}\ , \qquad t \in [1,2]\ , \\ y(1) &= -\frac{1}{2}\ . \end{aligned} \tag{2}$$

(a) Develop Taylor's method of order 2 to solve the IVP (2). Please write your derivations in detail.

Hint: In other words, find the derivative of the right-hand side of the ODE in (2) (which represents the second derivative $y''(t)$ of the exact solution), and write expressions for the values of w_0 and w_{i+1} (where w_{i+1} should be expressed in terms of the values of w_i, t_i and the stepsize h.

(b) Use the MATLAB code `taylor2nd.m` from the class web-site and a MATLAB file that will supply the right-hand side of the ODE and the first derivative of the right-hand side of the ODE with respect to t, to solve the IVP (2) by using Taylor method of order 2 in order to find the value of $y(2)$, with $N = 10$, 100, 1000, and 10000. Record the numerical values of $y(2)$ obtained by using different N. Attach your MATLAB printout.

Hint: If, for example, I am using the Taylor method of order 2 to solve an IVP for the ODE $\frac{\mathrm{d}y}{\mathrm{d}t} = 1 + \frac{y}{t}$, then I have to compute the derivative with respect to t of the right-hand side: $\frac{\mathrm{d}}{\mathrm{d}t}\left(1 + \frac{y(t)}{t}\right) = \frac{1}{t}$. If I want to create a MATLAB function `fun1_and_fun1der` that takes the values of t and y and returns the right-hand side and its first derivative, then I can create a MATLAB file `fun1_and_fun1der.m` that looks like this:

```
function [fun1, fun1der] = fun1_and_fun1der(t,y)
    fun1 = 1 + y/t;
    fun1der = 1/t;
```

(the semicolon at the end of a line of code means that MATLAB will not print the output of executing this line).

(c) Plot in MATLAB the logarithm of the error, $|y(2)_{\text{exact}} - y(2)_{\text{approx}}|$, versus the logarithm of the stepsize h. Find the slope of the straight line through the points on your graph, and discuss how this value compares with the theoretical prediction. The exact solution of the IVP (2) is

$$y(t)_{\text{exact}} = \tfrac{1}{2t}\left[\sqrt{3}\tan\left(\tfrac{\sqrt{3}}{2}\ln|t|\right) - 1\right]\ . \tag{3}$$

Plotting in MATLAB is very easy. To plot, say, the points (x_j, y_j) (for $j = 1,2,3,4,5$), where $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5) = (4,5,6,7,8)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4, y_5) = (12,11,12,10,9)$, you can type the following MATLAB commands:

```
x = [4 5 6 7 8]
y = [12 11 12 10 9]
plot(x,y)
```

If, instead of `plot(x,y)` you type `plot(x,y,'-*')`, the data will be represented by stars connected by straight lines. Type `help plot` for more information. To plot the data on a loglog plot, type

```
loglog(x,y,'-*')
```

For more information, see

```
https://www.mathworks.com/help/matlab/ref/loglog.html
```

Problem 3. Consider the same IVP as in Problem 3.

(a) Develop Taylor's method of order 3 to solve the IVP (2).
Hint: Here is what I obtained for the second derivative of $f(t, y(t))$:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t, y(t)) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(y(t)^2 + \frac{1}{t^2}\right) = 6y^4(t) + \frac{8y^2(t)}{t^2} - \frac{4y(t)}{t^3} + \frac{8}{t^4}\ .$$

(b) Write a MATLAB code called

```
yourfirstname_yourfamilyname_taylor3rd.m
```

(where `yourfirstname` and `yourfamilyname` should be replaced with your actual names) that solves IVPs for ODEs by using Taylor's method of order 3. Perhaps the easiest way to do this will be to take `taylor2nd.m` and to modify it (the modification will be really minor). Do not forget to the change first line of the file to

```
function [wi,ti] = yourfirstname_yourfamilyname_taylor3rd(RHS,t0,x0,tf,N)
```

(the name of the file and the name of the function in it must match, otherwise MATLAB will complain). Write a MATLAB file that returns the right-hand side of the ODE and all the derivatives that are needed by `taylor3rd.m`. Attach to your homework printouts of the code `yourfirstname_yourfamilyname_taylor3rd.m` and the code that returns the right-hand side.

(c) Run `yourfirstname_yourfamilyname_taylor3rd.m` to solve the IVP (2) with $N = 10$, 100, 1000, and 10000. Attach the printout of the run. Record the numerical values of $y(2)$ obtained by using different N in a table.

(d) Plot in MATLAB the logarithm of the error, $|y(2)_{\text{exact}} - y(2)_{\text{approx}}|$, versus the logarithm of the stepsize h. Find the slope of the straight line through the points on your graph, and discuss how this value compares with the theoretical prediction. The exact solution of the IVP is given in (3).