

Problem 1.

- (a) Show that the function
- $y(t)$
- defined implicitly by the equation

$$y(t)^3 - t \sin(y(t)) + t^2 - 1 = 0 \quad (1)$$

satisfies the IVP

$$\begin{aligned} \frac{dy}{dt} &= \frac{\sin y - 2t}{3y^2 - t \cos y}, & t \geq 0, \\ y(0) &= 1. \end{aligned} \quad (2)$$

Hint: To derive the differential equation for $y(t)$, differentiate (1) with respect to t .

- (b) Find the numerical value of $y(\frac{1}{2})$ by solving the equation (1) using Newton's method. In other words, derive Newton's functional iteration and apply it to equation (1) by using the Matlab code `newton.m` available on the class web-site. Use some reasonable value of the tolerance, say, 10^{-12} (recall that the accuracy of Matlab is about 10^{-16}). Attach the Matlab output of running the code `newton.m` in verbose mode.

Hint: If you have forgotten how to run a Matlab code, look at the instructions for running the code `bisection.m` in the materials accompanying Homework 3.

- (c) In the materials accompanying this homework on the class web-site, you will find the Matlab codes `euler.m` and `rhs.m` needed in this part of the problem, as well as instructions how to use them.

Use the Matlab code `euler.m` to solve the IVP (2) and find $y(\frac{1}{2})$. Do it with $N = 10, 100, 1000, 10000,$ and 100000 (which corresponds to stepsize $h = 0.05, 0.005, 0.0005, 0.00005$ and 0.000005 , respectively). In a table put the values of N , the corresponding values of $y(\frac{1}{2})_{\text{approx}}$ obtained by running `euler.m`, as well as the absolute errors $|y(\frac{1}{2})_{\text{exact}} - y(\frac{1}{2})_{\text{approx}}|$, where $y(\frac{1}{2})_{\text{exact}}$ is the value found in part (a) by using Newton's method (using small enough tolerance, i.e., 10^{-12}).

- (d) Plot by hand or using some software the logarithm of the error, $|y(\frac{1}{2})_{\text{exact}} - y(\frac{1}{2})_{\text{approx}}|$, versus the logarithm of the stepsize h . Find the slope of the approximate straight line that goes through these points. How does the value of this slope match with the theoretical prediction for the value of the error of Euler's method?

Also, you can use natural logarithms or logarithms base 10, or any other base to plot the results (but use the same base for both axes!) – this is not going to change the slope of the approximate straight line.

Problem 2. The so-called *Fresnel integrals*, $S(t)$ and $C(t)$, are two functions, named after the French physicist Augustin-Jean Fresnel (1788–1827), who used them in the theory of wave optics. They are defined as follows:

$$S(t) := \int_0^x \sin(z^2) dz , \quad C(t) := \int_0^x \cos(z^2) dz .$$

One can find the values of these integrals for any given value of t by simply calculating the value of the integral using, say, Simpson’s formula. Another way to do this is to write each of these functions as a solution of an IVPs for an ODE and the solve this IVP. Your goal in this problem will be to compute the value of $S(1)$.

- (a) Construct an IVP

$$\begin{aligned} \frac{dy}{dt} &= f(t, y) , & t \in [a, b] , \\ y(a) &= \alpha \end{aligned}$$

such that the value of the solution $y(t)$ of this IVP at $t = 1$ be equal to $S(1)$. In other words, find the function $f(t, y)$ and some appropriate numerical values of a , b , and α . Explain briefly your choices.

- (b) Use Euler’s method with $N = 10, 100, 1000, 10000$, and 100000 , to find $S(1)$. Record your values and the values of the absolute errors in each case. The exact value of $S(1)$ is

$$S(1)_{\text{exact}} = 0.31026830172338110180815242316539650757450938883 \dots .$$

Remark: Note that the Mathematica functions `FresnelS` and `FresnelC` are defined differently than the definition above, namely,

$$S(t) = \sqrt{\frac{\pi}{2}} \text{FresnelS} \left[\sqrt{\frac{2}{\pi}} t \right] , \quad C(t) = \sqrt{\frac{\pi}{2}} \text{FresnelC} \left[\sqrt{\frac{2}{\pi}} t \right] .$$

Problem 3. Consider the IVP

$$\begin{aligned} \frac{dy}{dt} &= (t + y)^2 , & t \in \left[0, \frac{\pi}{6} \right] , \\ y(0) &= 2 - \sqrt{3} . \end{aligned} \tag{3}$$

- (a) Develop Taylor’s method of order 2 to solve the IVP (3).
- (b) In the materials accompanying this homework on the class web-site, you will find the Matlab codes `taylor2.m` and `tworkers.m` that can be used to apply Taylor method of

order 2 in order to solve the IVP considered in Section 5.2, Example 1 (page 259) and in Section 5.3, Example 1 (page 268).

Write your own Matlab function `twoders.m` and use it together with `taylor2.m` to solve the IVP (3) by using Taylor method of order 2. Run `taylor2.m` with $N = 10, 100, 1000,$ and $10000,$ and record the numerical values of $y(\frac{\pi}{6})$.

Note that when you write a row vector in Matlab, as in the code `twoders.m`, the space is understood as a separator between the entries (i.e., between the components of the vector). For example,

```
twoD = [y-t^2+1 y-t^2+1-2*t];
```

defines `twoD` as the 2-dimensional vector $(y - t^2 + 1, y - t^2 + 1 - 2t)$; if you type instead

```
twoD = [y-t^2+1 y-t^2 +1-2*t];
```

then Matlab will think that `twoD` is the 3-dimensional vector $(y - t^2 + 1, y - t^2, 1 - 2t)$.

Please attach a printout of your Matlab codes!

- (c) Plot by hand or using some software the logarithm of the error, $|y(\frac{\pi}{6})_{\text{exact}} - y(\frac{\pi}{6})_{\text{approx}}|$, versus the logarithm of the stepsize h . Find the slope of the straight line through the points on your graph, and discuss how this value compares with the theoretical prediction. The exact solution of the IVP (3) is

$$y(t)_{\text{exact}} = \tan\left(x + \frac{\pi}{12}\right) - x .$$