MATH 4073 Homework 10 Due 12/01/09

Problem 1.

(a) Show that the function $y(t)$ defined implicitly by the equation

$$y(t)^2 - 2t\mathrm{e}^{-y(t)} - 4t^2 - 1 = 0 \tag{1}$$

satisfies the IVP

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{4t + \mathrm{e}^{-y}}{y + t\mathrm{e}^{-y}} \ , \qquad t \geq 0 \ , \\ y(0) &= 1 \ . \end{aligned} \tag{2}$$

Hint: To derive the differential equation for $y(t)$, differentiate (1) with respect to t.

(b) Find the numerical value of $y(\frac{1}{2})$ by solving the equation (1) using Newton's method. In other words, derive Newton's functional iteration and apply it to equation (1) by using the Matlab code `newton.m` available on the class web-site. Use some reasonable value of the tolerance, say, 10^{-12} (recall that the accuracy of Matlab is about 10^{-16}).

Hint: If you have forgotten how to run a Matlab code, look at the instructions for running the code `bisection.m` in the materials accompanying Homework 3.

(c) In the materials accompanying this homework on the class web-site, you will find the Matlab codes `euler.m` and `rhs.m` needed in this part of the problem, as well as instructions how to use them.

Use the Matlab code `euler.m` to solve the IVP (2) and find $y(\frac{1}{2})$. Do it with $N = 10$, 100, 1000, 10000, and 100000 (which corresponds to stepsize $h = 0.05$, 0.005, 0.0005, 0.00005 and 0.000005, respectively). In a table put the values of N, the corresponding values of $y(\frac{1}{2})_{\text{approx}}$ obtained by running `euler.m`, as well as the absolute errors $|y(\frac{1}{2})_{\text{exact}} - y(\frac{1}{2})_{\text{approx}}|$, where $y(\frac{1}{2})_{\text{exact}}$ is the value found in part (a) by using Newton's method (using small enough tolerance, i.e., 10^{-12}).

(d) Plot by hand or using some software the logarithm of the error, $|y(\frac{1}{2})_{\text{exact}} - y(\frac{1}{2})_{\text{approx}}|$, versus the logarithm of the stepsize h. Find the slope of the approximate straight line that goes through these points. How does the value of this slope match with the theoretical prediction for the value of the error of Euler's method?

Also, you can use natural logarithms or logarithms base 10, or any other base to plot the results (but use the same base for both axes!) – this is not going to change the slope of the approximate straight line.

Problem 2. As you already know, the *error function* is defined as

$$\operatorname{erf}(t) := \frac{2}{\sqrt{\pi}} \int_0^t \mathrm{e}^{-x^2}\, \mathrm{d}x \ .$$

In this problem you will find the value of erf(1) by using a method different from the one you used in Homework 9.

(a) Write an IVP for the function $\operatorname{erf}(t)$:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= f(t,y) \ , \qquad t \in [a,b] \ , \\ y(a) &= \alpha \ . \end{aligned}$$

In other words, find the numerical values of a, b, and α, and the function $f(t,y)$.

(b) Use Euler's method with $N = 10$, 100, 1000, 10000, and 100000, to find erf(1).

Compare your results with the exact value, which, as you know, is,

$\operatorname{erf}(1)_{\text{exact}} = 0.842700792949714869341220635082609259296066997966302908459 9\ldots$.

Problem 3. Consider the IVP

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= y^2 + \frac{1}{t^2} \ , \qquad t \in [1,2] \ , \\ y(1) &= -\frac{1}{2} \ . \end{aligned} \tag{3}$$

(a) Develop Taylor's method of order 3 to solve the IVP (3).

(b) In the materials accompanying this homework on the class web-site, you will find the Matlab codes `taylor2.m` and `twoders.m` needed in this part of the problem, as well as instructions how to use them for the IVP treated in Section 5.2, Example 1 (page 259) and in Section 5.3, Example 1 (page 270).

Write a Matlab code called `taylor3.m` and a Matlab function `threeders.m` to solve the IVP (3) by using Taylor method of order 3. Run `taylor3.m` with $N = 10$, 100, 1000, and 10000, and record the numerical values of $y(2)$.

Please attach a printout of your Matlab codes!

(c) Plot by hand or using some software the logarithm of the error, $|y(2)_{\text{exact}} - y(2)_{\text{approx}}|$, versus the logarithm of the stepsize h. Find the slope of the straight line through the points on your graph, and discuss how this value compares with the theoretical prediction. The exact solution of the IVP (3) is

$$y(t)_{\text{exact}} = \frac{1}{2t}\left[\sqrt{3}\tan\left(\frac{\sqrt{3}}{2}\ln|t|\right) - 1\right] \ .$$