MATH 3423 Homework 10 Due Thu, 11/21/19

Problem 1. Determine the geometric meaning of the operators A, B, and C acting on $\mathbb{R}^2$, if they are represented by the following matrices:

$$\underline{\underline{A}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad \underline{\underline{B}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad \underline{\underline{C}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} .$$

Hint: Take an arbitrary vector in $\mathbb{R}^2$, say $\mathbf{u} = \binom{2}{1}$, draw $\mathbf{u}$ and at the products $\underline{\underline{A}}\mathbf{u}$, $\underline{\underline{B}}\mathbf{u}$, and $\underline{\underline{C}}\mathbf{u}$ in $\mathbb{R}^2$, and the geometric meaning of the corresponding operators will be transparent.

Problem 2.

(a) Directly from the definition of product of matrices, show that $(\underline{\underline{A}}\,\underline{\underline{B}})^T = \underline{\underline{B}}^T \underline{\underline{A}}^T$.

(b) Directly from the definition of orthogonality of matrices (for the case of Euclidean inner product), i.e., $\underline{\underline{A}}^T \underline{\underline{A}} = \underline{\underline{I}}$, prove that the product of two orthogonal matrices is orthogonal.

(c)

Problem 3. Let the linear operator in the 2-dimensional vector space V with basis $\mathbf{f}_1$, $\mathbf{f}_2$, be defined by

$$\mathsf{A}\mathbf{f}_1 = -\mathbf{f}_1 + 4\mathbf{f}_2 ,$$
$$\mathsf{A}\mathbf{f}_2 = \mathbf{f}_1 + 2\mathbf{f}_2 .$$

(a) Write down the matrix $\underline{\underline{A}}$ of the linear operator A in the basis $\mathbf{f}_1$, $\mathbf{f}_2$.

(b) Compute the eigenvalues and the eigenvectors of this matrix.

Remark: In class we wrote $\underline{\underline{A}}$ and found that $\lambda_1 = -2$, $\lambda_2 = 3$, $\mathbf{u}_1 = \binom{1}{-1}$. Here you only have to find an eigenvector $\mathbf{u}_2$. As you know, $\mathbf{u}_2$ is not uniquely defined; choose $\mathbf{u}_2$ it in such a way that its first component be equal to 1.

(c) Now you know that

$$\mathbf{u}_1 = \mathbf{f}_1 - \mathbf{f}_2 ,$$
$$\mathbf{u}_2 = \mathbf{f}_1 + (?)\,\mathbf{f}_2 .$$

Express the original basis vectors $\mathbf{f}_1 = \binom{1}{0}$ and $\mathbf{f}_2 = \binom{0}{1}$ in terms of the eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$. (Do not use any “canned” formulas, just do the obvious calculations.)

(d) Use the relations

$$\mathbf{u}_1 = \mathbf{f}_1 - \mathbf{f}_2 \ ,$$
$$\mathbf{u}_2 = \mathbf{f}_1 + (?)\,\mathbf{f}_2$$

obtained in part (b), and the relations

$$\mathbf{f}_1 = \frac{4}{5}\,\mathbf{u}_1 + (?)\,\mathbf{u}_2 \ ,$$
$$\mathbf{f}_2 = (?)\,\mathbf{u}_1 + \frac{1}{5}\,\mathbf{u}_2$$

obtained in part (c), as well as the definition of the linear operator A in the statement of the problem (i.e., the action of A on the basis $\mathbf{f}_1$, $\mathbf{f}_2$), to express $\mathsf{A}\mathbf{u}_1$ and $\mathsf{A}\mathbf{u}_2$ in terms of $\mathbf{u}_1$ and $\mathbf{u}_2$. At the end the result will be totally obvious, but I want to see your detailed calculations.

(e) Since the eigenvalues of the matrix $\underline{\underline{A}}$ are real and distinct, a theorem guarantees that the eigenvectors of the linear operator A form a basis of the linear space V. Let $\underline{\underline{\widetilde{A}}} = (\widetilde{a}_{ij})$ be the matrix of the linear operator A in the basis $\mathbf{u}_1$, $\mathbf{u}_2$, i.e., $\mathsf{A}\mathbf{u}_j = \sum_{i=1}^{2} \widetilde{a}_{ij}\,\mathbf{u}_i$.

Find the entries $\widetilde{a}_{ij}$ of the matrix $\underline{\underline{\widetilde{A}}}$.

Remark: The result will be obvious, but I want to see all calculations that I am asking you to perform.

Problem 4. Consider the linear constant coefficient system

$$\begin{aligned} x_1'(t) &= x_1(t) + 2x_2(t) \\ x_2'(t) &= 2x_1(t) + x_2(t) \ . \end{aligned} \tag{1}$$

(a) Write the system (1) in the form $\mathbf{x}'(t) = \underline{\underline{A}}\mathbf{x}(t)$. Note that $\underline{\underline{A}}$ is a symmetric matrix.

(b) Find the eigenvectors and the normalized eigenvectors of the symmetric matrix $\underline{\underline{A}}$.

(c) The general theory claims that the eigenvalues of a symmetric matrix are real, and the eigenvectors corresponding to different eigenvalues are orthogonal (with respect to the Euclidean inner product, $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T\mathbf{v} = \sum_i u_i v_i$). Check that these properties hold for the matrix $\underline{\underline{A}}$ from parts (a) and (b).

(d) Normalize the eigenvectors of the matrix $\underline{\underline{A}}$, and write the matrix $\underline{\underline{S}}$ whose columns are the normalized eigenvectors of $\underline{\underline{A}}$.

(e) Show by a direct calculation that the matrix $\underline{\underline{S}}$ from part (d) is orthogonal (with respect to the Euclidean inner product), i.e., that $\underline{\underline{S}}^T\underline{\underline{S}} = \underline{\underline{I}}$.

(f) Find $\underline{\underline{S}}^{-1}$. You can answer this question without doing any calculations, but please explain what properties you are using.

(g) Find $\underline{\underline{D}} = \underline{\underline{S}}^{-1}\underline{\underline{A}}\,\underline{\underline{S}}$ and compute $\mathrm{e}^{t\underline{\underline{D}}}$.

(h) Use your results from parts (d)–(g) to compute $\mathrm{e}^{t\underline{\underline{A}}}$.

(i) Use your results to write down the general solution $\mathbf{x}(t)$ of the system (1).

(j) Use your result from part (i) to find the particular solution of the system (1) that satisfies the initial condition $\mathbf{x}(0) = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

Problem 5. Solve the linear constant coefficient system (1) from the previous problem with initial condition $\mathbf{x}(0) = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ by using that, if all eigenvalues λ_j of the matrix $\underline{\underline{A}}$ are distinct, then the general solution of the system $\mathbf{x}'(t) = \underline{\underline{A}}\mathbf{x}(t)$ is given by

$$\mathbf{x}(t) = \sum_{j=1}^{n} C_j \mathrm{e}^{\lambda_j t}\mathbf{u}_j \ ,$$

where $\mathbf{u}_j$ are the corresponding eigenvectors (not necessarily normalized).

Problem 6. Determine the eigenvalues and eigenvectors of the matrix $\underline{\underline{A}} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$. How many linearly independent eigenvectors does it have?

Remark: This problem shows the trouble one may encounter in the case of repeated eigenvalues.

Food for Thought Problem 1. (This problem does not need to be turned in.)

Express the coefficients of the characteristic polynomial, $\det(\underline{\underline{A}} - \lambda\underline{\underline{I}})$, of the matrix $\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in terms of $\det A$ and $\operatorname{tr} A$.

Food for thought: The eigenvalues of an operator A should not depend on the choice of basis (because their definition did not require a choice of basis). On the other hand, the eigenvalues are roots of the characteristic equation $\det(\underline{\underline{A}} - \lambda\underline{\underline{I}}) = 0$, which depends on the choice of basis in V (because in different bases the matrix $\underline{\underline{A}}$ of the linear operator A looks different). We know from the handout *Change of basis in a linear space* (linked at Lecture 23) that, if the change of basis is defined by the (invertible) matrix $\underline{\underline{A}}$, then the matrix $\underline{\underline{\widetilde{A}}}$ of the operator A in the new basis is related to the matrix $\underline{\underline{A}}$ of the operator in the old basis by $\underline{\underline{\widetilde{A}}} = \underline{\underline{C}}\,\underline{\underline{A}}\,\underline{\underline{C}}^{-1}$. This poses the question whether the characteristic polynomials $\det(\underline{\underline{A}} - \lambda\underline{\underline{I}})$ and $\det(\underline{\underline{\widetilde{A}}} - \lambda\underline{\underline{I}})$

are the same (as functions of λ). Recalling the property $\det(\underline{\underline{A}}\,\underline{\underline{B}}) = \det(\underline{\underline{A}})\det(\underline{\underline{B}})$ (which also implies that $\det(\underline{\underline{A}}^{-1}) = (\det \underline{\underline{A}})^{-1}$), we obtain

$$
\begin{aligned}
\det(\widetilde{\underline{\underline{A}}} - \lambda \underline{\underline{I}}) &= \det(\underline{\underline{C}}\,\underline{\underline{A}}\,\underline{\underline{C}}^{-1} - \lambda \underline{\underline{I}}) = \det(\underline{\underline{C}}\,\underline{\underline{A}}\,\underline{\underline{C}}^{-1} - \lambda \underline{\underline{C}}\,\underline{\underline{I}}\,\underline{\underline{C}}^{-1}) \\
&= \det\left(\underline{\underline{C}}(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{\underline{C}}^{-1}\right) = \det(\underline{\underline{C}})\det(\underline{\underline{A}} - \lambda \underline{\underline{I}})\det(\underline{\underline{C}}^{-1}) = \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) \ ,
\end{aligned}
$$

therefore the characteristic equations for the matrix of the operator A does not depend on the basis.

One can also check that the determinant and the trace of the matrix of a linear operator A do not depend on the choice of basis. As you had to show in this problem, the characteristic equation has $\det \underline{\underline{A}}$ and $\operatorname{tr} \underline{\underline{A}}$ as coefficients. One can use the property of determinants to show that the determinant does not depend on the choice of basis:

$$
\det(\widetilde{\underline{\underline{A}}}) = \det(\underline{\underline{C}}\,\underline{\underline{A}}\,\underline{\underline{C}}^{-1}) = \det \underline{\underline{C}}\ \det \underline{\underline{A}}\ \det(\underline{\underline{C}}^{-1}) = \det \underline{\underline{C}}\ \det \underline{\underline{A}}\,(\det \underline{\underline{C}})^{-1} = \det \underline{\underline{A}} \ .
$$

As for the trace, one can easily prove that

$$
\operatorname{tr}\left(\underline{\underline{A}}\,\underline{\underline{B}}\,\underline{\underline{C}}\,\underline{\underline{D}}\right) = \operatorname{tr}\left(\underline{\underline{B}}\,\underline{\underline{C}}\,\underline{\underline{D}}\,\underline{\underline{A}}\right) = \operatorname{tr}\left(\underline{\underline{C}}\,\underline{\underline{D}}\,\underline{\underline{A}}\,\underline{\underline{B}}\right) = \operatorname{tr}\left(\underline{\underline{D}}\,\underline{\underline{A}}\,\underline{\underline{B}}\,\underline{\underline{C}}\right)
$$

(cyclic permutation of the product the matrices in the trace; analogous formula holds for the trace of the product of any number of matrices, not only four matrices as in this equality). Therefore

$$
\operatorname{tr}\left(\underline{\underline{C}}\,\underline{\underline{A}}\,\underline{\underline{C}}^{-1}\right) = \operatorname{tr}\left(\underline{\underline{A}}\,\underline{\underline{C}}^{-1}\,\underline{\underline{C}}\right) = \operatorname{tr}\left(\underline{\underline{A}}\,\underline{\underline{I}}\right) = \operatorname{tr}\left(\underline{\underline{A}}\right) \ .
$$

All this provides another proof that the eigenvalues of a 2×2 matrix does *not* depend on the choice of a basis.