MATH 3423   Homework 10   Due Mon, 11/25/13

Problem 1. In this problem you will prove that if the Lagrangian does not depend explicitly on time, i.e., if $L = L(q(t), \dot{q}(t))$ (as opposed to the general case $L = L(q(t), \dot{q}(t), t)$), then

$$\dot{q}\,\frac{\partial L}{\partial \dot{q}} - L = \text{const} . \tag{1}$$

(a) The total derivative of $L = L(q(t), \dot{q}(t), t)$ with respect to t is

$$\frac{\mathrm{d}}{\mathrm{d}t} L(q(t), \dot{q}(t), t) = \frac{\partial L}{\partial q}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{\partial L}{\partial \dot{q}}\frac{\mathrm{d}\dot{q}}{\mathrm{d}t} + \frac{\partial L}{\partial t} = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{\partial L}{\partial t} .$$

How does this expression change when L does not depend explicitly on time, i.e., $L = L(q(t), \dot{q}(t))$?

(b) Take the time derivative of $\dot{q}\,\dfrac{\partial L}{\partial \dot{q}}$ and use the Euler-Lagrange equation to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}\frac{\partial L}{\partial \dot{q}}\right) = \dot{q}\frac{\partial L}{\partial q} + \ddot{q}\frac{\partial L}{\partial \dot{q}} .$$

(c) From your results in parts (a) and (b) derive that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right) = 0 ,$$

which is equivalent to the desired relation (1) (why?).

Problem 2. Derive the Euler-Lagrange equation for the action functional

$$I[q] = \int_{t_1}^{t_2} t\sqrt{1 - \dot{q}^2}\,\mathrm{d}t .$$

Problem 3. Derive the Euler-Lagrange equation for the action functional

$$I[q] = \int_{t_1}^{t_2} \left(t\dot{q}^2 - q\dot{q} + q\right)\mathrm{d}t .$$

Problem 4.

(a) Find the extremum of the functional $I[q] = \displaystyle\int_{t_1}^{t_2} L(q_1(t), q_2(t), \dot{q}_1(t), \dot{q}_2(t))\,\mathrm{d}t$, where $L(q_1(t), q_2(t), \dot{q}_1(t), \dot{q}_2(t)) = \sqrt{\dot{q}_1^2 + \dot{q}_2^2}$.

(b) Your result from part (a) has a very simple geometric meaning – what is it?

Problem 5. Derive Euler-Lagrange equations describing the motion of a point particle in a gravitational field if the particle is constrained to lie on a circle of radius a in a fixed vertical plane. Clearly, the particle has only one degree of freedom (for example, its position can be described by the angle $\theta(t)$ between the vertical direction and the line connecting the particle with the center of the circle). The Lagrangian is equal to the difference between the kinetic and the potential energy of the particle.

Problem 6. In this problem you will prove the fact that "the Lagrangian is defined only up to an additive total time derivative of any function of the coordinate and the time". The meaning of this statement is made precise below.

Let

$$\widetilde{L}(q(t), \dot{q}(t), t) = L(q(t), \dot{q}(t), t) + \frac{\mathrm{d}}{\mathrm{d}t} f(q(t), t) ,$$

where $f(q, t)$ is an arbitrary function of the coordinate q and the time t.

(a) Show that the action functionals corresponding to L and $\widetilde{L}$ differ by an additive constant, namely,

$$\int_{t_1}^{t_2} \widetilde{L}(q(t), \dot{q}(t), t)\, \mathrm{d}t = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t)\, \mathrm{d}t + f(q^{(2)}, t_2) - f(q^{(1)}, t_1) ,$$

where the end values of $q(t)$ are kept fixed, $q(t_1) = q^{(1)}$, $q(t_2) = q^{(2)}$ (as in the derivation of the Euler-Lagrange equations). Since $f(q^{(2)}, t_2)$ and $f(q^{(1)}, t_1)$ do not change when $q(t)$ is replaced by $q(t) + \eta(t)$ (where the variation, $\eta(t)$, vanishes at t_1 and t_2), the above relation implies that the Euler-Lagrange equations coming from L are the same as those coming from $\widetilde{L}$.

(b) Now you will obtain the same result from the Euler-Lagrange equations. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t} f(q(t), t) = \frac{\partial f}{\partial q}(q(t), t)\, \dot{q}(t) + \frac{\partial f}{\partial t}(q(t), t) ,$$

and show that

$$\frac{\partial \widetilde{L}}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \widetilde{L}}{\partial \dot{q}} = \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} .$$

Please give all your calculations in detail.