MATH 3413 **Homework 10** **Due Mon, Nov 29, 2010**

Problem 1. In all parts of the problem below, you can use without deriving the following solutions of the heat equation $u_t(x,t) = \alpha^2 u_{xx}(x,t)$, $x \in [0,L]$, $t \geq 0$, with appropriate boundary conditions; the first expression is for zero temperature at both boundaries (homogeneous Dirichlet BCs), and the second is for zero heat flux at both boundaries (homogeneous Neumann BCs):

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} b_n \exp\left\{-\left(\frac{n\pi\alpha}{L}\right)^2 t\right\} \sin\frac{n\pi x}{L} \ , \\ u(x,t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp\left\{-\left(\frac{n\pi\alpha}{L}\right)^2 t\right\} \cos\frac{n\pi x}{L} \ . \end{aligned}$$

(a) Solve the Dirichlet BVP below, find the asymptotic temperature, $u_\infty(x) := \lim_{t\to\infty} u(x,t)$, and explain why the expression you obtained for $u_\infty(x)$ is physically obvious.

$$\begin{aligned} &u_t = 9u_{xx} \ , \qquad x \in [0,\pi] \ , \quad t \geq 0 \ , \\ &u(0,t) = 0 \ , \quad u(\pi,t) = 0 \ , \\ &u(x,0) = 4\sin 2x + 7\sin 5x \ . \end{aligned}$$

(b) Derive and use a trigonometric relation to solve the following Dirichlet BVP:

$$\begin{aligned} &u_t = u_{xx} \ , \qquad x \in [0,\pi] \ , \quad t \geq 0 \ , \\ &u(0,t) = 0 \ , \quad u(\pi,t) = 0 \ , \\ &u(x,0) = 4\sin 4x \cos 2x \ . \end{aligned}$$

(c) Solve the Neumann BVP below, find the asymptotic temperature, $u_\infty(x) := \lim_{t\to\infty} u(x,t)$, and explain why the expression you obtained for $u_\infty(x)$ is physically obvious.

$$\begin{aligned} &u_t = 9u_{xx} \ , \qquad x \in [0,5] \ , \quad t \geq 0 \ , \\ &u_x(0,t) = 0 \ , \quad u_x(5,t) = 0 \ , \\ &u(x,0) = 7 + 6\cos 2\pi x \ . \end{aligned}$$

(d) Solve the Neumann BVP below. You may use the results of Problem 4 of Section 9.3 without deriving them.

$$\begin{aligned} &u_t = 9u_{xx} \ , \qquad x \in [0,2] \ , \quad t \geq 0 \ , \\ &u_x(0,t) = 0 \ , \quad u_x(2,t) = 0 \ , \\ &u(x,0) = f(x) := \begin{cases} x & \text{for } x \in [0,1] \ , \\ 2-x & \text{for } x \in [1,2] \ . \end{cases} \end{aligned}$$

Problem 2. Consider the following BVP with non-homogeneous Dirichlet BCs:

$$\begin{aligned} &u_t = 9u_{xx} \ , \qquad x \in [0, \pi] \ , \quad t \geq 0 \ , \\ &u(0,t) = 0 \ , \quad u(\pi, t) = 5 \ , \\ &u(x, 0) = 0 \ . \end{aligned}$$

(a) Set $u(x,t) = \ell(x) + v(x,t)$, where $\ell(x)$ is a linear function of x that satisfies the conditions $\ell(0) = 0$ and $\ell(\pi) = 5$ (compare these with the boundary conditions that the function u satisfies). Clearly, there is only one such linear function ℓ, namely, $\ell(x) = \frac{5}{\pi}x$. Derive the BVP satisfied by the function $v(x,t)$ – you will obtain a BVP with homogeneous (i.e., zero) Dirichlet BCs. Be careful – the PDE for v may be different than the PDE for u, and the IC for v will certainly be different from the one for u.

(b) Solve the BVP for v derived in part (a). You again may use the expressions for the solutions of BVPs for the heat equation given in Problem 1 (without deriving them). Also, you may use the fact that the sine Fourier series of the function $f(x) = x$ for $x \in [0, L]$ is

$$\frac{2L}{\pi}\left(\sin\frac{\pi x}{L} - \frac{1}{2}\sin\frac{2\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} - \frac{1}{4}\sin\frac{4\pi x}{L} + \cdots\right) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\frac{n\pi x}{L}$$

(this expression is derived in Example 1 on pages 600–601 of the book).

(c) Having solved part (b), write down the solution $u(x,t)$ of the original BVP.

Problem 3. In this problem you can use without deriving that the solution of the wave equation $u_{xx}(x,t) - \frac{1}{c^2}u_{tt}(x,t) = 0$, $x \in [0, L]$, $t \geq 0$, with homogeneous Dirichlet BCs $u(0,t) = 0$, $u(L,t) = 0$ has the form

$$u(x,t) \quad = \quad \sum_{n=1}^{\infty} T_n(t) \sin\frac{n\pi x}{L} \ .$$

The functions $T_n(t)$ satisfy the ODEs

$$T_n''(t) + \left(\frac{n\pi c}{L}\right)^2 T_n(t) = 0 \ , \qquad t \geq 0 \ ,$$

so their general form is

$$T_n(t) = A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L} \ .$$

The constants A_n and B_n can be found from the initial conditions, $u(x,0) = f(x)$ (initial position of the spring) and $u_t(x,0) = g(x)$ (initial velocity of the spring).

Solve the BVP

$$
\begin{aligned}
&u_{xx} - \frac{1}{9}u_{tt} = 0 \ , \qquad x \in [0,\pi] \ , \quad t \geq 0 \ , \\
&u(0,t) = 0 \ , \quad u(\pi,t) = 0 \ , \\
&u(x,0) = 4\sin 2x \ , \qquad u_t(x,0) = 15 \sin 5x \ .
\end{aligned}
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