MATH 5763 **Homework 1** **Due Thu, 1/24/2019**

Problem 1. An apparatus consists of three modules, labeled 1, 2, and 3. For the apparatus to be working, module 1 must be working, and at least one of the other two modules, 2 and 3, must be working. Let E_i stands for the event "module $\#i$ is working", and let $P(E_i) = p_i$. The events E_i are independent (i.e., module $\#i$ is working independently of whether the other modules are working).

Parts (a) and (b) of this problem are very similar to the problem about the electric circuit solved in class.

(a) Express the event $W :=$ "the apparatus is working" in terms of the events E_1, E_2, and E_3.

(b) Use your answer in part (a) to find the probability that the apparatus is working. Please indicate where you use the independence of the events E_1, E_2, and E_3.

(c) Find the probability that the apparatus is working if module #3 has burned (i.e., is *not* working)?

Remark: You can answer this question very easily by a simple reasoning involving almost no math, or can use the formal definition of conditional probability. You do *not* need to write both methods, one is enough.

(d) Show that the probability that module #3 has burned (i.e., is *not* working) if the apparatus is working is $\mathbb{P}(E_3^c|W) = \dfrac{p_2(1-p_3)}{p_2+p_3-p_2p_3}$.

Problem 2. Olivia and Nathan go target shooting together. Both shoot at the same target at the same time. Suppose that Olivia hits the target with probability $p_{\mathrm{O}} = 0.7$, whereas Nathan, independently, hits the target with probability $p_{\mathrm{N}} = 0.4$.

(a) Determine the probability that the target is hit (by at least one bullet).

(b) Given that the target is hit, show that the probability that Nathan hit it is $\frac{0.4}{0.82}$.

(c) Find the probability that *exactly* one shot hits the target.

(d) Given that *exactly* one shot hits the target, show that the probability that it was Nathan's shot is $\frac{0.12}{0.54}$.

Remark: This problem becomes really transparent if you draw a Venn diagram with the events $O = \{\text{Olivia hits the target}\}$ and $N = \{\text{Nathan hits the target}\}$. In your solution you have to use the formal definition of conditional probability, but at the end you can use the Venn diagram to check your answers.

Problem 3. Let X_1, …, X_n be n independent random variables, and let $F_{X_n}(x)$ be the cumulative distribution function of the random variable X_n, defined as the probability of the event $\{X_n \le x\}$:

$$F_{X_n}(x) = \mathbb{P}(X_n \le x) \ .$$

Let Y be a random variable that is equal to the maximum of the random variables $X_1, \ldots, X_n$: $Y = \max\{X_1, \ldots, X_n\}$.

(a) Let x be a real number. Explain why the event $\{Y \le x\}$ can be written as the intersection of the events $\{X_1 \le x\}, \ldots, \{X_n \le x\}$, i.e., $\{Y \le x\} = \bigcap_{j=1}^{n} \{X_j \le x\}$.

(b) Use the result of part (a) to show that the c.d.f. of Y is the product of the c.d.f.s of the random variables $X_1, \ldots, X_n$, i.e., $F_Y(x) = \prod_{j=1}^{n} F_{X_j}(x)$. Please indicate clearly where you have used the independence of $X_1, \ldots, X_n$.

Problem 4. Let X be a continuous RV uniformly distributed over the interval $[0, 1]$, i.e., the p.d.f. of X is

$$f_X(x) = \begin{cases} 1 & \text{if } x \in [0,1] \, , \\ 0 & \text{otherwise} \, . \end{cases}$$

Let X_1 and X_2 be independent continuous RVs modeled after the RV X, i.e., $f_{X_1}(x) = f_{X_2}(x) = f_X(x)$ for any $x \in \mathbb{R}$.

(a) Derive the expression for the c.d.f. $F_X(x)$ of the random variable X.

(b) Prove that the minimum, $X_{\min} = \min\{X_1, X_2\}$, of the RVs X_1 and X_2 has c.d.f.

$$F_{X_{\min}}(x) = 1 - [1 - F_X(x)]^2 \, .$$

Please point out at which point in your proof you used the independence of X_1 and X_2.

Hint: Use that $F_{X_{\min}}(x) = \mathbb{P}(X_{\min} \le x)$ and express the event $\{X_{\min} \le x\}$ in terms of the events $\{X_1 \le x\}$ and $\{X_2 \le x\}$.

(c) Use your result for the c.d.f. of $X_{\min}$ from part (b) to find the p.d.f. $f_{X_{\min}}(x)$ of $X_{\min}$.

(d) What can you say about the expectation of $X_{\min}$ *without doing any calculations*? (Not the exact value, just say *something reasonable*, and explain how you came to this conclusion.)

(e) Now compute the exact value of $\mathbb{E}[X_{\min}]$.

Problem 5. Let X be a continuous RV uniformly distributed over the interval $[0, 1]$ as in Problem 4. Define the continuous RV Y to be a function of the RV X defined as

$$Y = -\ln X \, .$$

(a) Find the interval of values where Y takes values. (Do not worry that X can take value 0 – this event occurs with probability 0, so you can ignore it. Think of X as taking values in the interval $(0, 1]$.)

(b) Directly from the definition of the c.d.f., $F_Y(y) = \mathbb{P}(Y \le y)$, of the RV Y, show that

$$F_Y(y) = \begin{cases} 0 & \text{if } \ y < 0 \ , \\ 1 - \mathrm{e}^{-y} & \text{if } \ y > 0 \end{cases}$$

(note that I did not write anything about $F_Y(0)$ – since Y is a continuous RV, $\mathbb{P}(Y = 0) = 0$, so that the value of $F_Y(0)$ is not important). You may that the following events are the same:

$$\{Y \le y\} = \{-\ln X \le y\} = \{\ln X \ge -y\} = \{X \ge \mathrm{e}^{-y}\} = \{X < \mathrm{e}^{-y}\}^c \ .$$

(c) Use your result from part (b) to find $f_Y(y)$.

(d) Now use the formula for the p.d.f. of a function, $Y = g(X)$, of a RV, namely

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) \right| \ ,$$

to find $f_Y(y)$ (of course, you should obtain the same result as in part (c)).

Food for Thought Problem 1.[1] Let A and B be events with $\mathbb{P}(A) = \frac{3}{4}$, and $\mathbb{P}(B) = \frac{1}{3}$. Convince me that

$$\frac{1}{12} \le \mathbb{P}(A \cap B) \le \frac{1}{3}$$

by a brief verbal explanation and/or a clear picture.

Food for Thought Problem 2. Let A and B be events with $\mathbb{P}(A) = 0.3$ and $\mathbb{P}(B) = 0.4$. Find the conditional probability $\mathbb{P}(A|B)$ in the following cases:

(a) A and B are mutually exclusive (i.e., disjoint);

(b) $\mathbb{P}(A \cap B) = 0.1$;

(c) A implies B (i.e., every time A occurs, B also occurs).

Food for Thought Problem 3. Suppose that two fair dice – a red one and a green one – are rolled. Consider the events

$$\begin{aligned} A &= \text{"The red die shows an odd number."} \\ B &= \text{"The green die shows an odd number."} \\ C &= \text{"The sum of the numbers on the two dice is odd."} \end{aligned}$$

Show that the events A, B, and C are pairwise independent, but not independent.

[1] "Foot for Thought" problems are for you to think about, but they do *not* need to be turned in with the regular homework.