MATH 5763 **Homework 1** **Due Wed, 9/2/2015**

Problem 1. An apparatus consists of three modules, labeled 1, 2, and 3. For the apparatus to be working, module 1 must be working, and at least one of the other two modules, 2 and 3, must be working. Let E_i stands for the event "module $\#i$ is working", and let $P(E_i) = p_i$. The events E_i are independent (i.e., module $\#i$ is working independently of whether the other modules are working).

(a) Express the event $W :=$ "the apparatus is working" in terms of the events E_1, E_2, and E_3.

(b) Find the probability that the apparatus is working.

(c) Find the probability that the apparatus is working if module #3 has burned (i.e., is *not* working)?

(d) What is the probability that module #3 has burned (i.e., is *not* working), if the apparatus is working?

Problem 2. Amal and Chao go target shooting together. Both shoot at the same target at the same time. Suppose that Amal hits the target with probability $p_{\mathrm{A}} = 0.7$, whereas Chao, independently, hits the target with probability $p_{\mathrm{C}} = 0.4$.

(a) Determine the probability that the target is hit (by at least one bullet).

(b) Given that the target is hit, what is the probability that Chao hit it?

(c) Find the probability that *exactly* one shot hits the target.

(d) Given that *exactly* one shot hits the target, what is the probability that it was Chao's shot?

Problem 3. The cumulative distribution function F_X of the discrete random variable X is the following:

$$F_X(x) = \begin{cases} 0\ , & x < -1\ , \\ \frac{1}{4}\ , & -1 \le x < 2\ , \\ \frac{3}{4}\ , & 2 \le x < 3\ , \\ 1\ , & 3 \le x\ . \end{cases}$$

(a) Plot the graph of F_X.

(b) What are the values that the random variable X takes?

(c) Find the probability mass function p_X of the random variable X.

(d) The *expectation*, $\mathbb{E}[X]$, of the random variable X is defined as

$$\mathbb{E}[X] = \sum_k x_k\, p_X(x_k)\ ,$$

where the summation is over all values x_k that X can take. Find the value of $\mathbb{E}[X]$.

Problem 4. Let X_1, …, X_n be n independent random variables, and let $F_{X_n}(x)$ be the cumulative distribution function of the random variable X_n, defined as the probability of the event $\{X_n \le x\}$:

$$F_{X_n}(x) = \mathbb{P}(X_n \le x) \ .$$

Let Y be a random variable that is equal to the maximum of the random variables X_1, …, X_n:

$$Y = \max\{X_1, \ldots, X_n\} \ .$$

(a) Let x be a real number. Explain why the event $\{Y \le x\}$ can be written as the intersection of the events $\{X_1 \le x\}$, …, $\{X_n \le x\}$:

$$\{Y \le x\} = \bigcap_{j=1}^{n} \{X_j \le x\} \ .$$

(b) Use the result of part (a) to show that the c.d.f. of Y is the product of the c.d.f.s of the random variables X_1, …, X_n:

$$F_Y(x) = \prod_{j=1}^{n} F_{X_j}(x) \ .$$

Please indicate clearly where you have used the independence of X_1, …, X_n.

Food for Thought Problem 1.[1] Let A and B be events with $\mathbb{P}(A) = \frac{3}{4}$, and $\mathbb{P}(B) = \frac{1}{3}$. Convince me that

$$\frac{1}{12} \le \mathbb{P}(A \cap B) \le \frac{1}{3}$$

by a brief verbal explanation and/or a clear picture.

Food for Thought Problem 2. Suppose that two fair dice – a red one and a green one – are rolled. Consider the events

$$\begin{aligned} A &= \text{“The red die shows an odd number.”} \\ B &= \text{“The green die shows an odd number.”} \\ C &= \text{“The sum of the numbers on the two dice is odd.”} \end{aligned}$$

Show that the events A, B, and C are pairwise independent, but not independent.

[1] “Foot for Thought” problems are for you to think about, but they do not need to be turned in with the regular homework.