MATH 5483 Homework 1 Due Thu, Jan25

Problem 1. As we mentioned in class, one can define different norms in the same linear space. In this problem you will study different norms in $\mathbb{R}^2$. Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_1\mathbf{i} + u_2\mathbf{j} \in \mathbb{R}^2$.

(a) Define the norm $\|\mathbf{u}\|_2$ by

$$\|\mathbf{u}\|_2 := \sqrt{u_1^2 + u_2^2} \ .$$

Draw the unit disk in $\mathbb{R}^2$ in this norm, i.e., the set of vectors $\{\mathbf{v} \in \mathbb{R}^2 \ : \ \|\mathbf{v}\|_2 \leq 1\}$.

(b) Define the norm $\|\mathbf{u}\|_1$ by

$$\|\mathbf{u}\|_1 := |u_1| + |u_2| \ .$$

Draw the unit disk in $\mathbb{R}^2$ in this norm, i.e., the set of vectors $\{\mathbf{v} \in \mathbb{R}^2 \ : \ \|\mathbf{v}\|_1 \leq 1\}$.

(c) Define the norm $\|\mathbf{u}\|_\infty$ by

$$\|\mathbf{u}\|_\infty := \max\{|u_1|, |u_2|\} \ .$$

Draw the unit disk in $\mathbb{R}^2$ in this norm, i.e., the set of vectors $\{\mathbf{v} \in \mathbb{R}^2 \ : \ \|\mathbf{v}\|_\infty \leq 1\}$.

(d) Show that, for any $\mathbf{u} \in V$,

$$\|\mathbf{u}\|_\infty \leq \|\mathbf{u}\|_1 \leq 2\,\|\mathbf{u}\|_\infty \ .$$

Two norms $\|\mathbf{u}\|_a$ and $\|\mathbf{u}\|_b$ on the same linear space are said to be *equivalent* if there exist positive constants C and C' such that $C\,\|\mathbf{u}\|_a \leq \|\mathbf{u}\|_b \leq C'\,\|\mathbf{u}\|_a$ for any vector $\mathbf{u}$. So what you just proved means that the norms $\|\mathbf{u}\|_\infty$ and $\|\mathbf{u}\|_1$ are equivalent.

(e) Show that the norms $\|\mathbf{u}\|_\infty$ and $\|\mathbf{u}\|_2$ are equivalent (you have to find the constants C and C' yourselves).

(f) Use parts (d) and (e) to prove that the norms $\|\mathbf{u}\|_1$ and $\|\mathbf{u}\|_2$ are equivalent. (This won't require any calculations!)

Problem 2. In finite-dimensional spaces, all norms are equivalent, but this is not the case in infinitely-dimensional spaces. Let ℓ_1, ℓ_2 and ℓ_3 be the linear spaces of all infinite sequences $\mathbf{u} = (u_1, u_2, \ldots)$ for which the corresponding norms,

$$\|\mathbf{u}\|_1 = \sum_{j\in\mathbb{N}} |u_j| \ , \qquad \|\mathbf{u}\|_2 = \left(\sum_{j\in\mathbb{N}} |u_j|^2\right)^{1/2} \ , \qquad \|\mathbf{u}\|_\infty = \sup_{j\in\mathbb{N}} |u_j| \ ,$$

are finite.

(a) Give an example of a sequence in ℓ_∞ that is not in ℓ_1. (Explain in one sentence.)

(b) Give an example of a sequence in ℓ_∞ that is not in ℓ_2. (Explain in one sentence.)

(c) Give an example of a sequence in ℓ_2 that is not in ℓ_1. (Explain in one sentence.)

Problem 3.

Consider the linear space (over the real numbers) of polynomials of degree no greater than n and with real coefficients. Let us introduce this space with the inner product

$$(P, Q) = \int_a^b P(x)\, Q(x)\, w(x)\, \mathrm{d}x \ ,$$

and denote this inner-product linear space by $V_n(a, b; w(x))$.
We want to construct polynomials D_0, D_1, …, D_n satisfying the following conditions:

(i) the polynomial D_k is of degree k;

(ii) then coefficient of x^k in D_k is equal to 1 (such polynomials are called *monic*);

(iii) the polynomials D_0, D_1, D_2, …, D_n form an orthogonal basis in the space of polynomials $V_n(0, \infty; w(x) = \mathrm{e}^{-x})$.

In the solution of this problem the following identity will be handy:

$$\int_0^\infty x^k\, \mathrm{e}^{-x}\, \mathrm{d}x = k!$$

(where, by definition, $0! = 1$).

(a) Clearly, $D_0(x) = 1$ for each $x \in [0, \infty)$. Find the only monic polynomial D_1 of degree 1 that is orthogonal to D_0.

(b) Find the only monic quadratic polynomial D_2 that is orthogonal to both D_0 and D_1.

(c) Show that the polynomial $P(x) = x^2 + 3$ can be represented as a linear combination of the polynomials D_0, D_1 and D_2 as follows: $P = D_2 + 4D_1 + 5D_0$.

(d) Show by direct integration that $(D_0, D_0) = 1$, $(D_1, D_1) = 1$, $(D_2, D_2) = 4$.

(e) Find the orthogonal projection, $\mathrm{proj}_{D_0+2D_1} Q$, of the polynomial $Q(x) = x^2 + 3$ onto the " straight line"

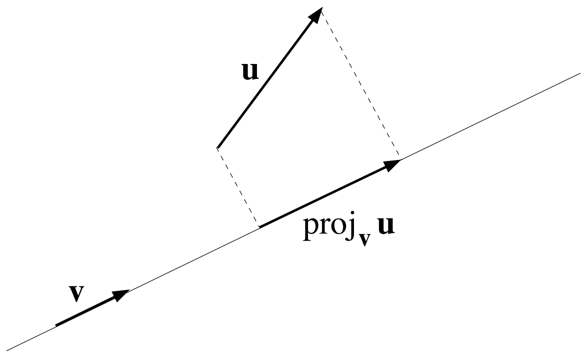
$$\ell := \{t(D_0 + 2D_1) \mid t \in \mathbb{R}\}$$

in the 3-dimensional inner product linear space $V_2(0, \infty; \mathrm{e}^{-x})$. If you have solved part (c), then finding this orthogonal projection should be easy.

Hint: If $\mathbf{u}$ and $\mathbf{v}$ are vectors in the inner product linear space V, then the orthogonal projection of the vector $\mathbf{u}$ onto the straight line in the direction of $\mathbf{v}$ is the vector

$$\mathrm{proj}_{\mathbf{v}}\mathbf{u} = \frac{(\mathbf{u}, \mathbf{v})}{(\mathbf{v}, \mathbf{v})}\,\mathbf{v}$$

– see the picture below.



(f) Finally, let $\tilde{D}_k := \mu_k D_k$, where $\mu_k > 0$ is a constant (depending on k) such that the norm,

$$\left\|\tilde{D}_k\right\| := \sqrt{\left(\tilde{D}_k, \tilde{D}_k\right)}\ ,$$

of the polynomial $\tilde{D}_k$ is 1. Find the explicit expressions for $\tilde{D}_0(x)$, $\tilde{D}_1(x)$, and $\tilde{D}_2(x)$.