MATH 4513 **Homework 1** **Due Fri, Sep 4, 2015**

Problem 1. [On the importance of thinking simply]

The distance between Norman, OK, and Waco, TX, is 270 miles, measured along I-35, which we assume to be a straight line (quite a realistic assumption). Two big trucks start driving on I-35 towards each other – one from Norman, the the other one from Waco. The one leaving from Norman is moving at 50 mph, the one leaving from Waco is moving at 40 mph.

At the same moment when the trucks start moving, an eagle starts flying from Norman towards Waco, at 70 mph. As soon as the eagle reaches the truck driving to the North, it turns around and starts flying to the North; as soon at it reaches the truck driving to the South, it turns around and starts flying to the South; the eagle keeps doing this until the two trucks meet, flying all the time at a speed of 70 mph.

Find the total distance traveled by the eagle from the beginning of its flight to the moment when the two trucks meet on the highway. Please explain your reasoning in detail. The solution can be very simple or quite complicated, depending on how you approach the problem.

Problem 2. [Linear stability analysis of an autonomous ODE]

Find the fixed points of the autonomous ODE

$$\dot{x} = f(x) = x(1-x)(2-x)$$

and analyze the stability of each fixed point (FP) by each of the following methods:

(a) by plotting the graph of $f(x) = x(1-x)(2-x)$ in the phase plane (i.e., the plane with x on the horizontal axis and $\dot{x}$ on the vertical axis), and observing the graph of f near each FP (plot arrows on the x-axis to indicate the direction of motion of the solution with time);

(b) by computing the derivatives of the function f at each of the three FPs.

Problem 3. [Stability of a degenerate fixed point of an autonomous ODE]

Consider the autonomous ODE

$$\dot{x} = f(x) \ ,$$

and assume that x^* is a FP (i.e., that $f(x^*) = 0$).

In class we analyzed in detail the behavior of the solution of this ODE if the initial condition is close to x^*. Namely, we solved the initial value problem (IVP)

$$\dot{x} = f(x) \ , \quad x(0) = x^* + \eta_0 \ , \quad 0 < |\eta_0| \ll 1 \ ; \tag{1}$$

the condition $0 < |\eta_0| \ll 1$ means that we assume that η_0 is nonzero and that $|\eta_0|$ is much smaller than 1, so that η_0^2 is negligible in comparison with $|\eta_0|$, etc. We assumed that the solution $x(t)$ of the IVP has the form

$$x(t) = x^* + \eta(t) \; , \qquad (2)$$

where $\eta(t)$ is small by absolute value (at least for some time), so that $\eta(t)^2$ and all higher powers of $\eta(t)$ are negligible in comparison with $|\eta(t)|$. For the function $\eta(t)$ we derived the following IVP:

$$\dot{\eta} = f'(x^*)\,\eta(t) + \tfrac{1}{2} f''(x^*)\,\eta(t)^2 + \cdots \; , \quad \eta(0) = \eta_0$$

(the fact that x^* is a FP implies that $f(x^*) = 0$). If $f'(x^*) \neq 0$, then this IVP simplifies to

$$\dot{\eta} = f'(x^*)\,\eta(t) \; , \quad \eta(0) = \eta_0 \; ,$$

whose solution is

$$\eta(t) = \eta_0\,\mathrm{e}^{At} \; , \quad A = f'(x^*) \; .$$

Clearly, if $f'(x^*) > 0$, $|\eta(t)|$ will grow exponentially with time, while if $f'(x^*) < 0$, $|\eta(t)|$ will decay exponentially to 0 with time. Recalling (2), we conclude that if $f'(x^*) > 0$, the FP x^* is unstable, and if $f'(x^*) < 0$, the FP x^* is stable. In this problem you will analyze the stability of x^* in the degenerate case $f'(x^*) = 0$.

(a) Assume that the first non-zero term in the Taylor expansion of $f(x)$ around x^* is the one with the second derivative:

$$f(x) = \tfrac{1}{2} f''(x^*)(x - x^*)^2 + \cdots \; .$$

Show that in this case the IVP for the function $\eta(t)$ (coming from the IVP (1) for the function $x(t)$ related to $\eta(t)$ by (2)) has the form

$$\dot{\eta} = B\eta(t)^2, \quad \eta(0) = \eta_0 \; . \qquad (3)$$

What is the constant B? Find the solution $\eta(t)$ of the IVP (3).

(b) Assume that $B < 0$ and determine whether the solution $\eta(t)$ grows unboundedly or decays to zero, in the case $\eta_0 < 0$, and separately in the case $\eta_0 > 0$ (please give detailed explanations). What can you conclude about the stability of the solution $x(t)$ in the case $B < 0$?

(c) Sketch the graph of f in a small vicinity of the FP x^*. Can you determine the stability of the FP x^* by simple reasoning without any computations? Please draw clear pictures (do not forget to label the axis!) and explain in detail.

(d) Do what you did in part (b), but in the case $B > 0$.

(e) Do what you did in part (d), but in the case $B > 0$.

Problem 4. [Number of roots of a 2-parameter family of quadratic equations]
Consider the 2-parameter family of quadratic equations

$$x^2 + 2ax + b = 0 , \tag{4}$$

where a and b are parameters (i.e., constants independent of x). Throughout this problem, by "roots" we mean "real roots."

(a) Write down the general formula for the roots of the quadratic equation (4); simplify it as much as you can.

(b) What is the condition on the parameters a and b for the equation (4) to have exactly one root? What is the condition for the equation (4) to have no roots? Why?

(c) What are the conditions on the parameters a and b for the equation (4) to have two roots of the same sign (i.e., either both positive or both negative)?

What are the conditions on the parameters a and b for the equation (4) to have two roots of opposite signs (one positive and one negative)? Give a clear justification of your answers.

(d) In the (a, b)-plane, plot the lines/regions found in parts (b) and (c) and clearly indicate them (i.e., write "1 root," "no roots," "2 roots same sign," "2 roots opposite signs" in your plot).

(e) Consider the autonomous ODE

$$\dot{x} = x^2 + 2ax + b . \tag{5}$$

Sketch the right-hand side of the equation (in other words, draw the graph of the function $f(x) = x^2 + ax + b$) and determine the number of FPs of (5) and their stability, in each of the following cases:

(e_1) the quadratic equation (4) has no solutions;

(e_2) the quadratic equation (4) has exactly one solution;

(e_3) the quadratic equation (4) has exactly two solutions.