

**Problem 1. [Semigroup property of the flow of an autonomous ODE]**

Consider the following IVP:

$$\begin{aligned}\frac{dx}{dt} &= x - x^2, & t > 0, \\ x(0) &= x_0 > 0.\end{aligned}\tag{1}$$

(a) Show that the solution of the IVP (1) is

$$\phi_t(x_0) = \frac{1}{1 + (x_0^{-1} - 1)e^{-t}}.$$

I want to see your detailed calculations. You may use the fact that

$$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}.$$

(b) Prove that the flow  $\phi_t$  from part (a) satisfies the semigroup condition,

$$\phi_t \circ \phi_s = \phi_{s+t}.$$

**Problem 2. [Solution of a constant-coefficient linear system as an exponential]**

If  $\mathbf{M}$  is a square  $m \times m$  matrix (i.e., a matrix of size  $m \times m$  with real or complex entries, one can define the *exponential* of  $\mathbf{M}$  as

$$\mathbf{e}^{\mathbf{M}} \equiv \exp \mathbf{M} := \sum_{j=0}^{\infty} \frac{1}{j!} \mathbf{M}^j,\tag{2}$$

where  $\mathbf{M}^0$  is by definition the identity matrix  $\mathbf{I}$ . It can be shown that this series converges for any square matrix  $\mathbf{M}$ .

Exponentials of matrices are useful for representing the solutions of initial-value problems for systems of linear ordinary differential coefficients with constant coefficients,

$$\begin{aligned}\frac{d\mathbf{x}}{dt} &= \mathbf{A} \mathbf{x}, & t > 0, \\ \mathbf{x}(0) &= \mathbf{x}^{(0)}.\end{aligned}\tag{3}$$

(a) Directly from the definition (2), show that  $\mathbf{M}\mathbf{e}^{\mathbf{M}} = \mathbf{e}^{\mathbf{M}}\mathbf{M}$  for any square matrix  $\mathbf{M}$ .

- (b) Let  $\mathbf{A}$  be a given  $m \times m$  matrix, and  $t$  be a real number. Then one can think of  $e^{\mathbf{A}t}$  as a function taking an argument from  $\mathbb{R}$  and having values in the  $m \times m$  matrices. Directly from (2), show that  $\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t}$  and  $e^{\mathbf{A}t}|_{t=0} = \mathbf{I}$ .

- (c) Use your result from part (b) to show that the solution of the initial-value problem (3) can be written as

$$\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{x}^{(0)} .$$

- (d) **[Only if you are taking the class as 5103; otherwise you get full credit]**

For any positive real numbers  $s$  and  $t$  show that  $e^{\mathbf{A}s}e^{\mathbf{A}t} = e^{\mathbf{A}(s+t)}$  and use this to show that  $\mathbf{x}(t+s) = e^{\mathbf{A}s}\mathbf{x}(t)$ . How can you interpret this result “physically”?

- (e) Directly from the definition (2), show that

$$e^{\mathbf{T}\mathbf{B}\mathbf{T}^{-1}} = \mathbf{T}e^{\mathbf{B}}\mathbf{T}^{-1} .$$

This representation is very convenient if  $e^{\mathbf{B}}$  is easy to compute. In particular, if  $\mathbf{B} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ , then  $e^{\mathbf{B}t} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$ .

- (f) Rewrite the linear system

$$\begin{aligned} \dot{x} &= 2x \\ \dot{y} &= 3x - y \end{aligned} \tag{4}$$

in a matrix form as  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ . If  $\mathbf{T} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$  with inverse  $\mathbf{T}^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ , find  $\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ .

- (g) Use your results from the previous part of this problem to write down  $e^{\mathbf{A}t}$  (where  $\mathbf{A}$  is the matrix from the right-hand side of (4)).
- (h) Use your result from part (h) to write down the solution of the initial-value problem consisting of the system (4) and the initial condition  $\mathbf{x}^{(0)} = \begin{pmatrix} x^{(0)} \\ y^{(0)} \end{pmatrix}$ . There is a line in  $\mathbb{R}^2$  such that if the initial point  $\mathbf{x}^{(0)}$  belongs to this line, then  $\phi_t(\mathbf{x}^{(0)})$  tends to the origin as  $t \rightarrow \infty$ , i.e.,  $\lim_{t \rightarrow \infty} \phi_t(\mathbf{x}^{(0)}) = \mathbf{0}$ . From the explicit expression for  $\phi_t(\mathbf{x}^{(0)})$  that you just obtained, find this line.

### Problem 3. [Poincaré map]

Consider the system

$$\dot{r} = r - r^2, \quad \dot{\theta} = 1, \tag{5}$$

where  $(r, \theta)$  are the polar coordinates in  $\mathbb{R}^2$ .

- (a) Find the solution  $(r(t), \theta(t))$  of (5), with initial conditions  $(r(0), \theta(0)) = (r_0, \theta_0)$ .  
*Hint:* You have already obtained the solution of the equation for  $r(t)$  in Problem 1; the solution of the equation for  $\theta(t)$  is trivial.
- (b) Let the Poincaré surface,  $\Sigma$ , be the positive  $x$ -axis (i.e., the set of points with  $\theta = 0$ ). We can use as coordinate on  $\Sigma$  the  $x$ -coordinate, i.e., the point  $(x, y) = (\xi, 0)$  on the positive  $x$ -axis (where  $(x, y)$  are the Cartesian coordinates of a point in  $\mathbb{R}^2$ ) is considered as a point in  $\Sigma$  with coordinate  $\xi > 0$ . Compute the Poincaré map from  $\Sigma$  to itself.
- (c) Show that the Poincaré map  $P : \Sigma \rightarrow \Sigma$  obtained in part (b) has a unique fixed point, i.e., a point  $\xi_* > 0$  such that  $P(\xi_*) = \xi_*$ .
- (d) Classify the stability of the fixed point of  $P$  found in part (c).  
*Hint:* You may find useful the fact that  $\frac{d}{d\xi} \frac{1}{1 + (\xi^{-1} - 1)e^{-2\pi}} = \frac{e^{-2\pi}}{\xi^2 [1 + (\xi^{-1} - 1)e^{-2\pi}]^2}$ .
- (e) Interpret your results from parts (c) and (d) in terms of the existence and stability of a periodic orbit of the system (5).