

Problem 1. [Semigroup property of the flow of an autonomous ODE]

Consider the following IVP:

$$\begin{aligned} \frac{dx}{dt} &= x - x^2, & t > 0, \\ x(0) &= x_0 > 0. \end{aligned} \tag{1}$$

(a) Show that the solution of the IVP (1) is

$$\phi_t(x_0) = \frac{1}{1 + (x_0^{-1} - 1)e^{-t}}.$$

I want to see your detailed calculations. You may use the fact that

$$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}.$$

(b) Prove that the flow ϕ_t from part (a) satisfies the semigroup condition,

$$\phi_t \circ \phi_s = \phi_{s+t}.$$

Problem 2. [Solution of a constant-coefficient linear system as an exponential]

If M is a square $m \times m$ matrix (i.e., a matrix of size $m \times m$ with real or complex entries, one can define the *exponential* of M as

$$e^M \equiv \exp M := \sum_{j=0}^{\infty} \frac{1}{j!} M^j, \tag{2}$$

where M^0 is by definition the identity matrix I . It can be shown that this series converges for any square matrix M .

Exponentials of matrices are useful for representing the solutions of initial-value problems for systems of linear ordinary differential coefficients with constant coefficients,

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= A \mathbf{x}, & t > 0, \\ \mathbf{x}(0) &= \mathbf{x}^{(0)}. \end{aligned} \tag{3}$$

(a) Directly from the definition (2), show that $M e^M = e^M M$ for any square matrix M .

(b) Let \mathbf{A} be a given $m \times m$ matrix, and t be a real number. Then one can think of $e^{\mathbf{A}t}$ as a function taking an argument from \mathbb{R} and having values in the $m \times m$ matrices. Directly from (2), show that $\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t}$ and $e^{\mathbf{A}t}|_{t=0} = \mathbf{I}$.

(c) Use your result from part (b) to show that the solution of the initial-value problem (3) can be written as

$$\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{x}^{(0)} .$$

(d) **[Only if you are taking the class as 5103; otherwise you get full credit]**

For any positive real numbers s and t show that $e^{\mathbf{A}s}e^{\mathbf{A}t} = e^{\mathbf{A}(s+t)}$ and use this to show that $\mathbf{x}(t+s) = e^{\mathbf{A}s}\mathbf{x}(t)$. How can you interpret this result “physically”?

(e) Directly from the definition (2), show that

$$e^{\mathbf{T}\mathbf{B}\mathbf{T}^{-1}} = \mathbf{T}e^{\mathbf{B}}\mathbf{T}^{-1} .$$

This representation is very convenient if $e^{\mathbf{B}}$ is easy to compute. In particular, if $\mathbf{B} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, then $e^{\mathbf{B}t} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$.

(f) Rewrite the linear system

$$\begin{aligned} \dot{x} &= 2x \\ \dot{y} &= 3x - y \end{aligned} \tag{4}$$

in a matrix form as $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. If $\mathbf{T} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ with inverse $\mathbf{T}^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$, find $\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$.

(g) Use your results from the previous part of this problem to write down $e^{\mathbf{A}t}$ (where \mathbf{A} is the matrix from the right-hand side of (4)).

(h) Use your result from part (g) to write down the solution of the initial-value problem consisting of the system (4) and the initial condition $\mathbf{x}^{(0)} = \begin{pmatrix} x^{(0)} \\ y^{(0)} \end{pmatrix}$. There is a line in \mathbb{R}^2 such that if the initial point $\mathbf{x}^{(0)}$ belongs to this line, then $\phi_t(\mathbf{x}^{(0)})$ tends to the origin as $t \rightarrow \infty$, i.e., $\lim_{t \rightarrow \infty} \phi_t(\mathbf{x}^{(0)}) = \mathbf{0}$. From the explicit expression for $\phi_t(\mathbf{x}^{(0)})$ that you just obtained, find this line.

Problem 3. [Poincaré map]

Consider the system

$$\dot{r} = r - r^2, \quad \dot{\theta} = 1, \tag{5}$$

where (r, θ) are the polar coordinates in \mathbb{R}^2 .

- (a) Find the solution $(r(t), \theta(t))$ of (5), with initial conditions $(r(0), \theta(0)) = (r_0, \theta_0)$.

Hint: You have already obtained the solution of the equation for $r(t)$ in Problem 1; the solution of the equation for $\theta(t)$ is trivial.

- (b) Let the Poincaré surface, Σ , be the positive x -axis (i.e., the set of points with $\theta = 0$). We can use as coordinate on Σ the x -coordinate, i.e., the point $(x, y) = (\xi, 0)$ on the positive x -axis (where (x, y) are the Cartesian coordinates of a point in \mathbb{R}^2) is considered as a point in Σ with coordinate $\xi > 0$. Compute the Poincaré map from Σ to itself.

- (c) Show that the Poincaré map $P : \Sigma \rightarrow \Sigma$ obtained in part (b) has a unique fixed point, i.e., a point $\xi_* > 0$ such that $P(\xi_*) = \xi_*$.

- (d) Classify the stability of the fixed point of P found in part (c).

Hint: You may find useful the fact that $\frac{d}{d\xi} \frac{1}{1 + (\xi^{-1} - 1)e^{-2\pi}} = \frac{e^{-2\pi}}{\xi^2 [1 + (\xi^{-1} - 1)e^{-2\pi}]^2}$.

- (e) Interpret your results from parts (c) and (d) in terms of the existence and stability of a periodic orbit of the system (5).