MATH 4193/5103 Homework 1 Due Tue, Feb 12, 2013

Problem 1. [Linear stability analysis of 1-dimensional systems]
Use linear stability analysis to classify the fixed points of the following systems:

(a) $\dot{x} = x(1-x)(2-x)$;

(b) $\dot{x} = 1 - \mathrm{e}^{-x^2}$.

Problem 2. [Potentials]
For each of the following systems, find and sketch the potential function $V(x)$, and use it to identify all the equilibrium points and determine their stability:

(a) $\dot{x} = \cos x$;

(b) $\dot{x} = 2 + \cos x$.

Problem 3. [Bifurcations in a two-parameter family]
In this problem you will study the bifurcations of the solutions of the two-parameter family of differential equations

$$\dot{x} = f_{a,b}(x) := -x^3 + 3ax + b \; . \tag{1}$$

(A) <u>**Case** $a > 0$</u>. Find the values of x where $f_{a,b}(x)$ reaches its extremal (i.e., minimal or maximal) values; call the smaller one x_- and the larger one x_+. Find the values of $f_{a,b}(x_-)$ (which is a local minimum of $f_{a,b}$) and $f_{a,b}(x_+)$ (which is a local maximum of $f_{a,b}$). Sketch the graph of $f_{a,b}$, and indicate on the graph the values you just found.

(A1) Find the range of values of the parameter b for which $f_{a,b}(x_+) < 0$. For b in this range do the following: (i) sketch the graph of $f_{a,b}$; (ii) plot the x-axis and indicate on it the equilibrium solutions (i.e., the fixed points) and their stability (as in Figure 2.1.1 of the book); (iii) plot the equilibrium solutions together with several other typical solutions on the (t, x)-plane (as in Figure 2.1.3).

(A2) Find the value of b for which $f_{a,b}(x_+) = 0$, and for this value of b repeat what you did in case (A1). What is the value of x_+ in this case? (Note that x_+ is a double root of $f_{a,b}(x) = 0$ for the value of b you just found, so $f_{a,b}(x_+) = 0$, $f'_{a,b}(x_+) = 0$.)

(A3) Find the range of b for which $f_{a,b}(x_-) < 0 < f_{a,b}(x_+)$, and for b in this range repeat what you did in case (A1).

(A4) Find the value of b for which $f_{a,b}(x_-) = 0$, and for this value of b repeat what you did in case (A1). What is the value of x_- in this case?

(A5) Find the range of b for which $f_{a,b}(x_-) > 0$, and for b in this range repeat what you did in case (A1).

(A6) For a fixed value $a > 0$, plot (by hand) the bifurcation diagram of (1), i.e., in the (b, x)-plane sketch the position of the fixed points as functions of b. Use solid line for the stable and dashed line for the unstable fixed points. Indicate all important values in your plot.

(B) **Case $a = 0$**. How many fixed points exist in this case? Find them explicitly, determine their stability, draw a sketch of the behavior of the solutions in the (t, x)-plane, as well as on the x-axis (as in Figure 2.1.1 of the book). Plot the bifurcation diagram in the (b, x)-plane.

(C) **Case $a < 0$**. Sketch the graph of $f_{a,b}(x)$ in this case. How many equilibrium solutions exist now? What about their stability? Sketch the behavior of the solutions as functions of the time; draw the x-axis with the fixed points on it, as in Figure 2.1.1 of the book. Sketch the bifurcation diagram in the (b, x)-plane.

(D) Now plot all your findings in (A), (B), and (C) about the number and type of equilibrium solutions (i.e., fixed points) in the (a, b)-plane. Plot the boundaries between the different regions, and in each region write the number of stable and unstable fixed points. Write the equations of the boundaries between the regions.

Problem 4. [Euler's method & MATLAB] **Only if you take the class as 5103!**

Use the MATLAB codes `euler1.m` and `rhs.m` to solve the initial-value problem

$$\begin{aligned} &\dot{x} = x - t^2 + 1 \ , \quad t \in [0, 2] \ , \\ &x(0) = 0.5 \ , \end{aligned} \tag{2}$$

whose exact solution is $x_{\text{exact}}(t) = (t + 1)^2 - \frac{1}{2}\,\mathrm{e}^t$.

Let $x^{(N)}_{\text{approx}}(2)$ stand for the approximate value of $x(2)$ obtained by running `euler1.m` and `rhs.m` with N steps, i.e., by running

```
euler1(@rhs,0.0,2.0,0.5,N)
```

where N takes values 10, 100, 1000, 10000, and 100000.

Let us call "error" the absolute value of the difference between the approximate and the exact values of $x(2)$, i.e.,

$$E_N := |x^{(N)}_{\text{approx}}(2) - x_{\text{exact}}(2)| \ .$$

Generally, E_N decreases with the stepsize, $h_N := \frac{b-a}{N} = \frac{2}{N}$, as $O((h_N)^k)$ for some $k > 0$:

$$E_N = O((h_N)^k) \ , \qquad \text{i.e.,} \ \ E_N \approx C\,(h_N)^k \ \text{ for small } h_N \ .$$

You have to find k "experimentally" for the Euler's method by studying the behavior of E_N for the initial problem (2). To this end, you can plot (by hand or using some software)

$\log E_N$ as a function of $\log h_N$ for the five values of N given above. Taking logarithms of $E_N \approx C\,(h_N)^k$, we obtain

$$\log E_N \approx \log C + k \log h_N \ ,$$

from which one can obtain k as the slope of the approximately straight line going through the points $(\log h_N, \log E_N)$. Find k "experimentally" by relying more on the larger N.

The MATLAB codes `euler1.m` and `rhs.m`, instructions how to run them, and several MATLAB tutorials are available on the class web-site. The 16-page tutorial by Peter Blossey is a good starting point for learning MATLAB.

"Food for Thought" Problem 5.[1] **[Big O, little o]**

Let $I = (a, b)$ be an open interval containing zero, and $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be two functions defined on I and taking values in $\mathbb{R}$ (the set of real numbers).

We say that $f(x)$ is big O of $g(x)$ (as x goes to 0), and write $f(x) = O(g(x))$ (or simply $f = O(g)$) if there exists a positive constants δ and M such that

$$|f(x)| \le M|g(x)| \quad \text{for } |x| < \delta \ . \tag{3}$$

One can show that, if the ratio of $|f(x)|$ and $|g(x)|$ tends to a finite limit as $x \to 0$, i.e., if $\lim\limits_{x\to 0} \dfrac{|f(x)|}{|g(x)|} < \infty$, then $f(x) = O(g(x))$.

We say that $f(x)$ is little o of $g(x)$ (as x goes to 0), and write $f = o(g)$ if

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = 0 \ . \tag{4}$$

We will be particularly interested in the case when $g(x) = x^\alpha$ for some constant $\alpha \in \mathbb{R}$.

(a) Directly from (3) show that, if $f = O(g)$ and $r = O(g)$, then $f + r = O(g)$ and $f \cdot r = O(g)$.

(b) Use (3) and (4) to show that, if $f = O(g)$ and $r = o(g)$, then $f \cdot g = o(g)$.

(c) Find the largest α such that $\sin^3 x = O(x^\alpha)$.

(d) If β is not zero or a positive integer, then it is not difficult to show that the Taylor series of z around 0 is

$$(1+z)^\beta = \sum_{k=a}^{\infty} \binom{\beta}{k} z^k \ ,$$

where $\binom{\beta}{k}$ is the generalized binomial coefficient, $\binom{\beta}{k} = \frac{\beta(\beta-1)\cdots(\beta-k+1)}{k!}$ (you do not need to prove this). Use this fact to show that

$$\sqrt{1+x^3} = 1 + \tfrac{1}{2}x^3 - \tfrac{1}{8}x^6 + O(x^9) \ .$$

[1] "Food for Thought" problems are not to be turned in, but you have to read them and think about them.