

**Problem 1. [Linear stability analysis of 1-dimensional systems]**

Use linear stability analysis to classify the fixed points of the following systems:

(a)  $\dot{x} = x(1-x)(2-x)$  ;

(b)  $\dot{x} = 1 - e^{-x^2}$  .

**Problem 2. [Potentials]**

For each of the following systems, find and sketch the potential function  $V(x)$ , and use it to identify all the equilibrium points and determine their stability:

(a)  $\dot{x} = \cos x$  ;

(b)  $\dot{x} = 2 + \cos x$  .

**Problem 3. [Bifurcations in a two-parameter family]**

In this problem you will study the bifurcations of the solutions of the two-parameter family of differential equations

$$\dot{x} = f_{a,b}(x) := -x^3 + 3ax + b . \quad (1)$$

- (A) **Case  $a > 0$ .** Find the values of  $x$  where  $f_{a,b}(x)$  reaches its extremal (i.e., minimal or maximal) values; call the smaller one  $x_-$  and the larger one  $x_+$ . Find the values of  $f_{a,b}(x_-)$  (which is a local minimum of  $f_{a,b}$ ) and  $f_{a,b}(x_+)$  (which is a local maximum of  $f_{a,b}$ ). Sketch the graph of  $f_{a,b}$ , and indicate on the graph the values you just found.
- (A1) Find the range of values of the parameter  $b$  for which  $f_{a,b}(x_+) < 0$ . For  $b$  in this range do the following: (i) sketch the graph of  $f_{a,b}$ ; (ii) plot the  $x$ -axis and indicate on it the equilibrium solutions (i.e., the fixed points) and their stability (as in Figure 2.1.1 of the book); (iii) plot the equilibrium solutions together with several other typical solutions on the  $(t, x)$ -plane (as in Figure 2.1.3).
- (A2) Find the value of  $b$  for which  $f_{a,b}(x_+) = 0$ , and for this value of  $b$  repeat what you did in case (A1). What is the value of  $x_+$  in this case? (Note that  $x_+$  is a double root of  $f_{a,b}(x) = 0$  for the value of  $b$  you just found, so  $f_{a,b}(x_+) = 0$ ,  $f'_{a,b}(x_+) = 0$ .)
- (A3) Find the range of  $b$  for which  $f_{a,b}(x_-) < 0 < f_{a,b}(x_+)$ , and for  $b$  in this range repeat what you did in case (A1).
- (A4) Find the value of  $b$  for which  $f_{a,b}(x_-) = 0$ , and for this value of  $b$  repeat what you did in case (A1). What is the value of  $x_-$  in this case?
- (A5) Find the range of  $b$  for which  $f_{a,b}(x_-) > 0$ , and for  $b$  in this range repeat what you did in case (A1).

- (A6) For a fixed value  $a > 0$ , plot (by hand) the bifurcation diagram of (1), i.e., in the  $(b, x)$ -plane sketch the position of the fixed points as functions of  $b$ . Use solid line for the stable and dashed line for the unstable fixed points. Indicate all important values in your plot.
- (B) **Case  $a = 0$ .** How many fixed points exist in this case? Find them explicitly, determine their stability, draw a sketch of the behavior of the solutions in the  $(t, x)$ -plane, as well as on the  $x$ -axis (as in Figure 2.1.1 of the book). Plot the bifurcation diagram in the  $(b, x)$ -plane.
- (C) **Case  $a < 0$ .** Sketch the graph of  $f_{a,b}(x)$  in this case. How many equilibrium solutions exist now? What about their stability? Sketch the behavior of the solutions as functions of the time; draw the  $x$ -axis with the fixed points on it, as in Figure 2.1.1 of the book. Sketch the bifurcation diagram in the  $(b, x)$ -plane.
- (D) Now plot all your findings in (A), (B), and (C) about the number and type of equilibrium solutions (i.e., fixed points) in the  $(a, b)$ -plane. Plot the boundaries between the different regions, and in each region write the number of stable and unstable fixed points. Write the equations of the boundaries between the regions.

**Problem 4. [Euler's method & MATLAB] Only if you take the class as 5103!**

Use the MATLAB codes `euler1.m` and `rhs.m` to solve the initial-value problem

$$\begin{aligned} \dot{x} &= x - t^2 + 1, & t \in [0, 2], \\ x(0) &= 0.5, \end{aligned} \tag{2}$$

whose exact solution is  $x_{\text{exact}}(t) = (t + 1)^2 - \frac{1}{2} e^t$ .

Let  $x_{\text{approx}}^{(N)}(2)$  stand for the approximate value of  $x(2)$  obtained by running `euler1.m` and `rhs.m` with  $N$  steps, i.e., by running

```
euler1(@rhs,0.0,2.0,0.5,N)
```

where  $N$  takes values 10, 100, 1000, 10000, and 100000.

Let us call “error” the absolute value of the difference between the approximate and the exact values of  $x(2)$ , i.e.,

$$E_N := |x_{\text{approx}}^{(N)}(2) - x_{\text{exact}}(2)|.$$

Generally,  $E_N$  decreases with the stepsize,  $h_N := \frac{2-a}{N} = \frac{2}{N}$ , as  $O((h_N)^k)$  for some  $k > 0$ :

$$E_N = O((h_N)^k), \quad \text{i.e., } E_N \approx C (h_N)^k \text{ for small } h_N.$$

You have to find  $k$  “experimentally” for the Euler’s method by studying the behavior of  $E_N$  for the initial problem (2). To this end, you can plot (by hand or using some software)

$\log E_N$  as a function of  $\log h_N$  for the five values of  $N$  given above. Taking logarithms of  $E_N \approx C (h_N)^k$ , we obtain

$$\log E_N \approx \log C + k \log h_N ,$$

from which one can obtain  $k$  as the slope of the approximately straight line going through the points  $(\log h_N, \log E_N)$ . Find  $k$  “experimentally” by relying more on the larger  $N$ .

The MATLAB codes `euler1.m` and `rhs.m`, instructions how to run them, and several MATLAB tutorials are available on the class web-site. The 16-page tutorial by Peter Blossey is a good starting point for learning MATLAB.

**“Food for Thought” Problem 5.<sup>1</sup> [Big  $O$ , little  $o$ ]**

Let  $I = (a, b)$  be an open interval containing zero, and  $f : I \rightarrow \mathbb{R}$  and  $g : I \rightarrow \mathbb{R}$  be two functions defined on  $I$  and taking values in  $\mathbb{R}$  (the set of real numbers).

We say that  $f(x)$  is big  $O$  of  $g(x)$  (as  $x$  goes to 0), and write  $f(x) = O(g(x))$  (or simply  $f = O(g)$ ) if there exists a positive constants  $\delta$  and  $M$  such that

$$|f(x)| \leq M|g(x)| \quad \text{for } |x| < \delta . \tag{3}$$

One can show that, if the ratio of  $|f(x)|$  and  $|g(x)|$  tends to a finite limit as  $x \rightarrow 0$ , i.e., if  $\lim_{x \rightarrow 0} \frac{|f(x)|}{|g(x)|} < \infty$ , then  $f(x) = O(g(x))$ .

We say that  $f(x)$  is little  $o$  of  $g(x)$  (as  $x$  goes to 0), and write  $f = o(g)$  if

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = 0 . \tag{4}$$

We will be particularly interested in the case when  $g(x) = x^\alpha$  for some constant  $\alpha \in \mathbb{R}$ .

- (a) Directly from (3) show that, if  $f = O(g)$  and  $r = O(g)$ , then  $f + r = O(g)$  and  $f \cdot r = O(g)$ .
- (b) Use (3) and (4) to show that, if  $f = O(g)$  and  $r = o(g)$ , then  $f \cdot g = o(g)$ .
- (c) Find the largest  $\alpha$  such that  $\sin^3 x = O(x^\alpha)$ .
- (d) If  $\beta$  is not zero or a positive integer, then it is not difficult to show that the Taylor series of  $z$  around 0 is

$$(1 + z)^\beta = \sum_{k=0}^{\infty} \binom{\beta}{k} z^k ,$$

where  $\binom{\beta}{k}$  is the generalized binomial coefficient,  $\binom{\beta}{k} = \frac{\beta(\beta-1)\dots(\beta-k+1)}{k!}$  (you do not need to prove this). Use this fact to show that

$$\sqrt{1 + x^3} = 1 + \frac{1}{2}x^3 - \frac{1}{8}x^6 + O(x^9) .$$

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<sup>1</sup>“Food for Thought” problems are not to be turned in, but you have to read them and think about them.