MATH 4193/5103 Homework 1 Due 2/4/20 (Tuesday)

Problem 1. [Semigroup property of the flow of an autonomous ODE]

Consider the following IVP:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x - x^2 , \qquad t > 0 , \\ x(0) &= x_0 > 0 . \end{aligned} \tag{1}$$

(a) Solve the IVP (1); its solution is

$$\phi_t(x_0) = \frac{1}{1 + \left(x_0^{-1} - 1\right)\mathrm{e}^{-t}} ,$$

but I want to see your detailed calculations. You may use the fact that

$$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$$

(easily obtained by the method of partial fractions, but you do *not* need to do this).

(b) Prove that the flow ϕ_t from part (a) satisfies the semigroup condition,

$$\phi_t \circ \phi_s = \phi_{s+t} .$$

Problem 2. [Solution of a constant-coefficient linear system as an exponential]

If M is a square $m \times m$ matrix (i.e., a matrix of size $m \times m$ with real or complex entries, one can define the *exponential* of M as

$$\mathrm{e}^{\mathsf{M}} \equiv \exp \mathsf{M} := \sum_{j=0}^{\infty} \frac{1}{j!} \mathsf{M}^j , \tag{2}$$

where M^0 is by definition the identity matrix I. It can be shown that this series converges for any square matrix M.

Exponentials of matrices are useful for representing the solutions of initial-value problems for systems of linear ordinary differential coefficients with constant coefficients,

$$\begin{aligned} \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} &= \mathsf{A}\,\mathbf{x} , \qquad t > 0 , \\ \mathbf{x}(0) &= \mathbf{x}^{(0)} . \end{aligned} \tag{3}$$

(a) Directly from the definition (2), show that $\mathsf{M}\mathrm{e}^{\mathsf{M}} = \mathrm{e}^{\mathsf{M}}\mathsf{M}$ for any square matrix M.

(b) Let A be a given $m \times m$ matrix, and t be a real number. Then one can think of $\mathrm{e}^{\mathsf{A}t}$ as a function taking an argument from $\mathbb{R}$ and having values in the $m \times m$ matrices. Directly from (2), show that $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathsf{A}t} = \mathsf{A}\mathrm{e}^{\mathsf{A}t}$ and $\mathrm{e}^{\mathsf{A}t}|_{t=0} = \mathsf{I}$.

(c) Use your result from part (b) to show that the solution of the initial-value problem (3) can be written as

$$\mathbf{x}(t) = \mathrm{e}^{\mathsf{A}t}\,\mathbf{x}^{(0)} \ .$$

(d) **[Only if you are taking the class as 5103; otherwise you get full credit]**
For any positive real numbers s and t show that $\mathrm{e}^{\mathsf{A}s}\mathrm{e}^{\mathsf{A}t} = \mathrm{e}^{\mathsf{A}(s+t)}$ and use this to show that $\mathbf{x}(t+s) = \mathrm{e}^{\mathsf{A}s}\mathbf{x}(t)$. How can you interpret this result "physically"?

(e) Directly from the definition (2), show that

$$\mathrm{e}^{\mathsf{TBT}^{-1}} = \mathsf{T}\mathrm{e}^{\mathsf{B}}\mathsf{T}^{-1} \ .$$

This representation is very convenient if e^{B} is easy to compute. In particular, if $\mathsf{B} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, then $\mathrm{e}^{\mathsf{B}t} = \begin{pmatrix} \mathrm{e}^{\lambda_1 t} & 0 \\ 0 & \mathrm{e}^{\lambda_2 t} \end{pmatrix}$.

(f) Rewrite the linear system

$$\begin{aligned} \dot{x} &= 2x \\ \dot{y} &= 3x - y \end{aligned} \tag{4}$$

in a matrix form as $\dot{\mathbf{x}} = \mathsf{A}\mathbf{x}$. If $\mathsf{T} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ with inverse $\mathsf{T}^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$, find $\mathsf{B} = \mathsf{T}^{-1}\mathsf{AT}$.

(g) Use your results from the previous part of this problem to write down $\mathrm{e}^{\mathsf{A}t}$ (where A is the matrix from the right-hand side of (4)).

(h) Use your result from part (h) to write down the solution of the initial-value problem consisting of the system (4) and the initial condition $\mathbf{x}^{(0)} = \begin{pmatrix} x^{(0)} \\ y^{(0)} \end{pmatrix}$. There is a line in $\mathbb{R}^2$ such that if the initial point $\mathbf{x}^{(0)}$ belongs to this line, then $\boldsymbol{\phi}_t(\mathbf{x}^{(0)})$ tends to the origin as $t \to \infty$, i.e., $\lim_{t\to\infty} \boldsymbol{\phi}_t(\mathbf{x}^{(0)}) = \mathbf{0}$. From the explicit expression for $\boldsymbol{\phi}_t(\mathbf{x}^{(0)})$ that you just obtained, find this line.

Problem 3. [Poincaré map]
Consider the system

$$\dot{r} = r - r^2 \ , \qquad \dot{\theta} = 1 \ , \tag{5}$$

where (r, θ) are the polar coordinates in $\mathbb{R}^2$.

(a) Find the solution $(r(t), \theta(t))$ of (5), with initial conditions $(r(0), \theta(0)) = (r_0, \theta_0)$.

Hint: You have already obtained the solution of the equation for $r(t)$ in Problem 1; the solution of the equation for $\theta(t)$ is trivial.

(b) Let the Poincaré surface, Σ, be the positive x-axis (i.e., the set of points with $\theta = 0$). We can use as coordinate on Σ the x-coordinate, i.e., the point $(x, y) = (\xi, 0)$ on the positive x-axis (where (x, y) are the Cartesian coordinates of a point in $\mathbb{R}^2$) is considered as a point in Σ with coordinate $\xi > 0$. Compute the Poincaré map from Σ to itself.

(c) Show that the Poincaré map $P : \Sigma \to \Sigma$ obtained in part (b) has a unique fixed point, i.e., a point $\xi_* > 0$ such that $P(\xi_*) = \xi_*$.

(d) Classify the stability of the fixed point of P found in part (c).

Hint: You may find useful the fact that $\dfrac{\mathrm{d}}{\mathrm{d}\xi} \dfrac{1}{1 + (\xi^{-1} - 1)\mathrm{e}^{-2\pi}} = \dfrac{\mathrm{e}^{-2\pi}}{\xi^2 \big[1 + (\xi^{-1} - 1)\mathrm{e}^{-2\pi}\big]^2}$.

(e) Interpret your results from parts (c) and (d) in terms of the existence and stability of a periodic orbit of the system (5).

"Food for Thought" Problem 1.[1] **[Taylor series, implicit differentiation]**

A very important tool that we will be using in this course is the Taylor expansion of a smooth function,

$$f(a+h) = f(a) + \frac{f'(a)}{1!}\,h + \frac{f''(a)}{2!}\,h^2 + \frac{f'''(a)}{3!}\,h^3 + \frac{f^{(4)}(a)}{4!}\,h^4 + \cdots$$

or, equivalently,

$$f(x) = f(a) + \frac{f'(a)}{1!}\,(x-a) + \frac{f''(a)}{2!}\,(x-a)^2 + \frac{f'''(a)}{3!}\,(x-a)^3 + \frac{f^{(4)}(a)}{4!}\,(x-a)^4 + \cdots \ .$$

The truncations $P_k(x)$ consisting of the terms of degree k and smaller,

$$P_1(x) = f(a) + \frac{f'(a)}{1!}\,(x-a) \ ,$$

$$P_2(x) = f(a) + \frac{f'(a)}{1!}\,(x-a) + \frac{f''(a)}{2!}\,(x-a)^2 \ , \ldots$$

are the best fitting polynomials to the function $f(x)$ at the point a.

Consider a function $y(x)$ defined implicitly by

$$x + y - y^3 = 0 \ . \qquad (6)$$

[1] "Food for Thought" problems are not to be turned in, but you have to read them and think about them.

Check that the point $(2, 1)$ belongs to the graph of the function. Use implicit differentiation to show that the straight line and parabola that fit best to the graph of $y(x)$ at the point $(2, 1)$ are given by

$$P_1(x) = 1 + \frac{1}{2}(x-2) \ , \qquad P_2(x) = 1 + \frac{1}{2}(x-2) - \frac{3}{8}(x-2)^2 \ . \tag{7}$$

The graphs of $y(x)$ and the truncations $P_1(x)$ and $P_2(x)$ are plotted in the figure below.

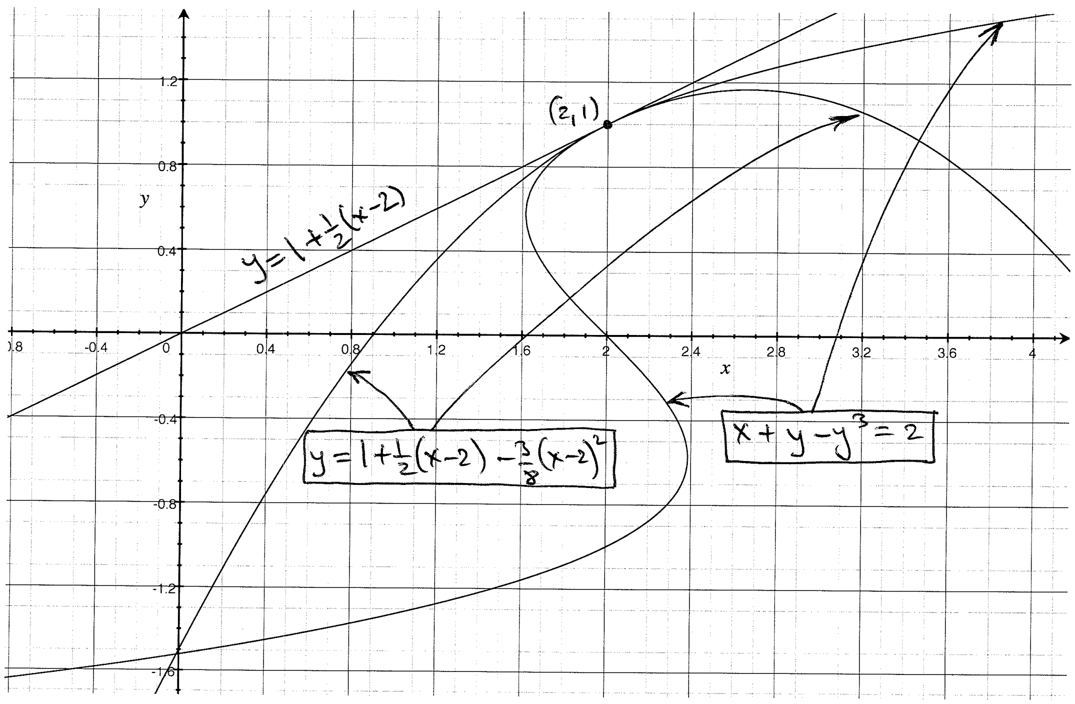

Figure 1: Graphs of the function defined implicitly by (6) and the straight line and parabola (given by (7)) that fit best to the graph of the function at the point $(2, 1)$.