MATH 4163 **Homework 1** **Due Thu, 01/25/2018**

Problem 1. The partial differential equation

$$u_t = u_{xx} + (u_x)^2 + \mathrm{e}^{-u}\sin x \tag{1}$$

where $u(x,t)$ is a function of two variables (and u_{xx} stands for the second partial derivative with respect to x, i.e., $u_{xx} := \frac{\partial^2 u}{\partial x^2}$), is very difficult to solve because it is nonlinear (due to the presence of the terms $(u_x)^2$ and e^{-u}). It, however, can be transformed to a linear equation by the so-called *Hopf-Cole transformation*,

$$u(x,t) = \ln v(x,t) \ , \tag{2}$$

where $v(x,t)$ is a new unknown function.

One can express the derivatives of the original function u in terms of the new function v and its derivatives. For example,

$$u_t(x,t) = \frac{\partial}{\partial t}\,u(x,t) = \frac{\partial}{\partial t}\,\ln v(x,t) = \frac{1}{v(x,t)}\,\frac{\partial}{\partial t}v(x,t) = \frac{v_t(x,t)}{v(x,t)}\ ,$$

where we have used the Chain Rule and the fact that $\dfrac{\mathrm{d}}{\mathrm{d}z}\ln z = \dfrac{1}{z}$.

(a) Use the Chain Rule to express $u_x(x,t)$ in terms of $v(x,t)$ and its derivatives.

(b) Use the Chain Rule again to find $u_{xx}(x,t)$ in terms of $v(x,t)$ and its derivatives.

(c) Use your results from parts (a) and (b) to show that the Hopf-Cole transformation (2) transforms the nonlinear equation (1) into a simpler equation (which does not contain nonlinear terms like the ones in (1)).

Problem 2. In this problem you will find the general solution of the third order PDE

$$u_{xyy}(x,y) = \mathrm{e}^{-x}\sin y \ . \tag{3}$$

(a) Integrate the PDE (3) with respect to x to obtain a second order PDE of the form $u_{yy}(x,y) = \ldots$. Do not forget that each integration introduces one arbitrary function.

(b) Integrate the PDE obtained in part (a) with respect to y to obtain a first order PDE of the form $u_y(x,y) = \ldots$.

(c) Integrate the PDE obtained in part (b) with respect to y to obtain the general solution $u(x,y)$ of the PDE (3).

Problem 3. Consider the first order PDE

$$\tan(x)\, u_x + y\, u_y = u \ , \quad u = u(x,y) \ , \tag{4}$$

on the semi-infinite strip $(0, \frac{\pi}{2}) \times (0, \infty)$ in $\mathbb{R}^2$ (i.e., for $x \in (0, \frac{\pi}{2})$, $y \in (0, \infty)$).

(a) Prove that the function

$$u(x,y) = y\, \varphi\Big(\frac{\sin x}{y}\Big) \ , \tag{5}$$

where φ is an arbitrary differentiable function of one variable, satisfies the PDE (4). In fact, the function $u(x,y)$ in (5) is the general solution of (4).

(b) Now impose the additional condition

$$u(t,t) = \sqrt{t^2 + \sin^2 t} \ , \quad t \in \big(0, \tfrac{\pi}{2}\big) \ , \tag{6}$$

on the general solution $u(x,y)$ (5) of the PDE (4). What is the concrete expression for the function $\varphi(t)$ in this case? Write down the solution $u(x,y)$ of the PDE (4) that satisfies the additional condition (6).

Problem 4. In this problem you will find the general solution of the first order PDE

$$u_x - \frac{y}{x}\, u_y + \frac{2}{3x}\, u = 0 \ , \quad u = u(x,y) \ , \tag{7}$$

by using an appropriate change of variables. As discussed in class, a change of variables from the "old" ones, (x,y), to the "new" ones, $(\widetilde{x}, \widetilde{y})$, is defined by a pair of functions, X and Y, as

$$\begin{aligned} x &= X(\widetilde{x}, \widetilde{y}) \ , \\ y &= Y(\widetilde{x}, \widetilde{y}) \ , \end{aligned} \tag{8}$$

or, equivalently, by the pair of functions, $\widetilde{X}$ and $\widetilde{Y}$, defining the inverse transform,

$$\begin{aligned} \widetilde{x} &= \widetilde{X}(x,y) \ , \\ \widetilde{y} &= \widetilde{Y}(x,y) \ . \end{aligned} \tag{9}$$

The two pairs of functions are related by

$$x = X\big(\widetilde{X}(x,y), \widetilde{Y}(x,y)\big) \ , \qquad y = Y\big(\widetilde{X}(x,y), \widetilde{Y}(x,y)\big)$$

or, equivalently, by

$$\widetilde{x} = \widetilde{X}\big(X(\widetilde{x},\widetilde{y}), Y(\widetilde{x},\widetilde{y})\big) \ , \qquad \widetilde{y} = \widetilde{Y}\big(X(\widetilde{x},\widetilde{y}), Y(\widetilde{x},\widetilde{y})\big) \ .$$

The "new" function, $\widetilde{u}(\widetilde{x},\widetilde{y})$, is defined by the requirement that the values of the functions $\widetilde{u}(\widetilde{x},\widetilde{y})$ and $u(x,y)$ at the corresponding points are the same. This is written as

$$\widetilde{u}\big(\widetilde{x},\widetilde{y}\big) := u\big(X(\widetilde{x},\widetilde{y}),Y(\widetilde{x},\widetilde{y})\big)$$

or, equivalently, as

$$u(x,y) =: \widetilde{u}\big(\widetilde{X}(x,y),\widetilde{Y}(x,y)\big)\ .$$

The relations between the derivatives of the functions u and $\widetilde{u}$ come from the Chain Rule. For example,

$$\begin{aligned} u_x(x,y) &= \frac{\partial}{\partial x}\,u(x,y) \\ &= \frac{\partial}{\partial x}\,\widetilde{u}\big(\widetilde{X}(x,y),\widetilde{Y}(x,y)\big) \\ &= \frac{\partial\widetilde{u}}{\partial\widetilde{x}}\big(\widetilde{X}(x,y),\widetilde{Y}(x,y)\cdot\frac{\partial\widetilde{X}}{\partial x}(x,y) + \frac{\partial\widetilde{u}}{\partial\widetilde{y}}\big(\widetilde{X}(x,y),\widetilde{Y}(x,y)\cdot\frac{\partial\widetilde{Y}}{\partial x}(x,y)\ , \end{aligned}$$

which is often written briefly as

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\,\widetilde{u}\big(\widetilde{X}(x,y),\widetilde{Y}(x,y)\big) = \frac{\partial\widetilde{u}}{\partial\widetilde{x}}\frac{\partial\widetilde{x}}{\partial x} + \frac{\partial\widetilde{u}}{\partial\widetilde{y}}\frac{\partial\widetilde{y}}{\partial x} = \widetilde{u}_{\widetilde{x}}\,\widetilde{x}_x + \widetilde{u}_{\widetilde{y}}\,\widetilde{y}_x\ .$$

(a) Consider the change of variables

$$x = \widetilde{x}\ , \qquad y = \frac{\widetilde{y}}{\widetilde{x}}\ .$$

Write down the inverse change of variables, i.e., express $\widetilde{x}$ and $\widetilde{y}$ in terms of x and y.

(b) Express u_x in terms of $\widetilde{u}_{\widetilde{x}}$ and $\widetilde{u}_{\widetilde{y}}$.

(c) Express u_y in terms of $\widetilde{u}_{\widetilde{x}}$ and $\widetilde{u}_{\widetilde{y}}$.

(d) Plug the expressions obtained in parts (b) and (c) in the PDE (7) to transform it into a PDE for $\widetilde{u}(\widetilde{x},\widetilde{y})$. There will be a cancellation, so that the equation for $\widetilde{u}(\widetilde{x},\widetilde{y})$ will contain only the derivative $\widetilde{u}_{\widetilde{x}}$, but not $\widetilde{u}_{\widetilde{y}}$, so that in part (e) you will be able to solve it as an ODE (in fact, a very simple separable ODE), treating $\widetilde{x}$ as a variable, and $\widetilde{y}$ as having a fixed value.

(e) Find the general solution of the simple PDE for $\widetilde{u}(\widetilde{x},\widetilde{y})$ derived in part (d). Your solution will contain one arbitrary function of one variable.

(f) Write the general solution $u(x,y)$ of the original PDE (7).