Problem 1. The partial differential equation

\[ u_t = u_{xx} + (u_x)^2 + e^{-u} \sin x \]  

(1)

where \( u(x,t) \) is a function of two variables (and \( u_{xx} \) stands for the second partial derivative with respect to \( x \), i.e., \( u_{xx} := \frac{\partial^2 u}{\partial x^2} \)), is very difficult to solve because it is nonlinear (due to the presence of the terms \((u_x)^2\) and \(e^{-u}\)). It, however, can be transformed to a linear equation by the so-called Hopf-Cole transformation,

\[ u(x,t) = \ln v(x,t), \]  

(2)

where \( v(x,t) \) is a new unknown function.

One can express the derivatives of the original function \( u \) in terms of the new function \( v \) and its derivatives. For example,

\[ u_t(x,t) = \frac{\partial}{\partial t} u(x,t) = \frac{1}{v(x,t)} \frac{\partial}{\partial t} v(x,t) = \frac{v_t(x,t)}{v(x,t)}, \]

where we have used the Chain Rule and the fact that \( \frac{d}{dz} \ln z = \frac{1}{z} \).

(a) Use the Chain Rule to express \( u_x(x,t) \) in terms of \( v(x,t) \) and its derivatives.

(b) Use the Chain Rule again to find \( u_{xx}(x,t) \) in terms of \( v(x,t) \) and its derivatives.

(c) Use your results from parts (a) and (b) to show that the Hopf-Cole transformation (2) transforms the nonlinear equation (1) into a simpler equation (which does not contain nonlinear terms like the ones in (1)).

Problem 2. In this problem you will find the general solution of the third order PDE

\[ u_{xyy}(x,y) = e^{-x} \sin y. \]  

(3)

(a) Integrate the PDE (3) with respect to \( x \) to obtain a second order PDE of the form \( u_{yy}(x,y) = \ldots \). Do not forget that each integration introduces one arbitrary function.

(b) Integrate the PDE obtained in part (a) with respect to \( y \) to obtain a first order PDE of the form \( u_y(x,y) = \ldots \).

(c) Integrate the PDE obtained in part (b) with respect to \( y \) to obtain the general solution \( u(x,y) \) of the PDE (3).
**Problem 3.** Consider the first order PDE

\[ \tan(x) u_x + y u_y = u , \quad u = u(x, y) , \quad (4) \]

on the semi-infinite strip \((0, \frac{\pi}{2}) \times (0, \infty)\) in \(\mathbb{R}^2\) (i.e., for \(x \in (0, \frac{\pi}{2})\), \(y \in (0, \infty)\)).

(a) Prove that the function

\[ u(x, y) = y \varphi \left( \frac{\sin x}{y} \right) , \quad (5) \]

where \(\varphi\) is an arbitrary differentiable function of one variable, satisfies the PDE (4). In fact, the function \(u(x, y)\) in (5) is the general solution of (4).

(b) Now impose the additional condition

\[ u(t, t) = \sqrt{t^2 + \sin^2 t} , \quad t \in (0, \frac{\pi}{2}) , \quad (6) \]

on the general solution \(u(x, y)\) (5) of the PDE (4). What is the concrete expression for the function \(\varphi(t)\) in this case? Write down the solution \(u(x, y)\) of the PDE (4) that satisfies the additional condition (6).

**Problem 4.** In this problem you will find the general solution of the first order PDE

\[ u_x - \frac{y}{x} u_y + \frac{2}{3x} u = 0 , \quad u = u(x, y) , \quad (7) \]

by using an appropriate change of variables. As discussed in class, a change of variables from the “old” ones, \((x, y)\), to the “new” ones, \((\bar{x}, \bar{y})\), is defined by a pair of functions, \(X\) and \(Y\), as

\[ x = X(\bar{x}, \bar{y}) , \quad y = Y(\bar{x}, \bar{y}) , \quad (8) \]

or, equivalently, by the pair of functions, \(\bar{X}\) and \(\bar{Y}\), defining the inverse transform,

\[ \bar{x} = \bar{X}(x, y) , \quad \bar{y} = \bar{Y}(x, y) . \quad (9) \]

The two pairs of functions are related by

\[ x = X(\bar{x}(x, y), \bar{Y}(x, y)) , \quad y = Y(\bar{X}(x, y), \bar{Y}(x, y)) \]

or, equivalently, by

\[ \bar{x} = \bar{X}(X(\bar{x}, \bar{y}), Y(\bar{x}, \bar{y})) , \quad \bar{y} = \bar{Y}(X(\bar{x}, \bar{y}), Y(\bar{x}, \bar{y})) . \]
The “new” function, $\tilde{u}(\tilde{x}, \tilde{y})$, is defined by the requirement that the values of the functions $\tilde{u}(\tilde{x}, \tilde{y})$ and $u(x, y)$ at the corresponding points are the same. This is written as

$$\tilde{u}(\tilde{x}, \tilde{y}) := u(X(\tilde{x}, \tilde{y}), Y(\tilde{x}, \tilde{y}))$$

or, equivalently, as

$$u(x, y) =: \tilde{u}(\tilde{X}(x, y), \tilde{Y}(x, y)).$$

The relations between the derivatives of the functions $u$ and $\tilde{u}$ come from the Chain Rule. For example,

$$u_x(x, y) = \frac{\partial}{\partial x} u(x, y) = \frac{\partial}{\partial \tilde{x}} \tilde{u}(\tilde{X}(\tilde{x}, \tilde{y}), \tilde{Y}(\tilde{x}, \tilde{y})) = \frac{\partial \tilde{u}}{\partial \tilde{x}}(\tilde{X}(\tilde{x}, \tilde{y}), \tilde{Y}(\tilde{x}, \tilde{y})) \cdot \frac{\partial \tilde{X}}{\partial x}(\tilde{x}, \tilde{y}) + \frac{\partial \tilde{u}}{\partial \tilde{y}}(\tilde{X}(\tilde{x}, \tilde{y}), \tilde{Y}(\tilde{x}, \tilde{y})) \cdot \frac{\partial \tilde{Y}}{\partial x}(x, y),$$

which is often written briefly as

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial \tilde{x}} \tilde{u}(\tilde{X}(x, y), \tilde{Y}(x, y)) = \frac{\partial \tilde{u}}{\partial \tilde{x}} \frac{\partial \tilde{X}}{\partial x} + \frac{\partial \tilde{u}}{\partial \tilde{y}} \frac{\partial \tilde{Y}}{\partial x} = \tilde{u}_x \tilde{x} + \tilde{u}_y \tilde{y}.$$

(a) Consider the change of variables

$$x = \tilde{x}, \quad y = \frac{\tilde{y}}{\tilde{x}}.$$

Write down the inverse change of variables, i.e., express $\tilde{x}$ and $\tilde{y}$ in terms of $x$ and $y$.

(b) Express $u_x$ in terms of $\tilde{u}_x$ and $\tilde{u}_y$.

(c) Express $u_y$ in terms of $\tilde{u}_x$ and $\tilde{u}_y$.

(d) Plug the expressions obtained in parts (b) and (c) in the PDE (7) to transform it into a PDE for $\tilde{u}(\tilde{x}, \tilde{y})$. There will be a cancellation, so that the equation for $\tilde{u}(\tilde{x}, \tilde{y})$ will contain only the derivative $\tilde{u}_x$, but not $\tilde{u}_y$, so that in part (e) you will be able to solve it as an ODE (in fact, a very simple separable ODE), treating $\tilde{x}$ as a variable, and $\tilde{y}$ as having a fixed value.

(e) Find the general solution of the simple PDE for $\tilde{u}(\tilde{x}, \tilde{y})$ derived in part (d). Your solution will contain one arbitrary function of one variable.

(f) Write the general solution $u(x, y)$ of the original PDE (7).