MATH 4163 **Homework 1** **Due Thu, Aug 30, 2012**

Problem 1. Two norms, $\| \ \|$ and $\| \ \|'$, on the same vector space V are said to be *equivalent* if there exist positive constants A and B such that

$$A\|\mathbf{u}\| \le \|\mathbf{u}\|' \le B\|\mathbf{u}\| \qquad \text{for any } \mathbf{u} \in V \ .$$

Consider the vector space $\mathbb{R}^n$ with the following norms defined on it:

$$\|\mathbf{u}\|_1 := \sum_{j=1}^{n} |u_j| \ , \qquad \|\mathbf{u}\|_2 := \left(\sum_{j=1}^{n} |u_j|^2 \right)^{1/2} \ , \qquad \|\mathbf{u}\|_\infty := \max_{1\le j\le n} |u_j| \ .$$

One can prove that the norms $\| \ \|_1$ and $\| \ \|_\infty$ on $\mathbb{R}^n$ are equivalent as follows: for an arbitrary vector $\mathbf{u} \in \mathbb{R}^n$ we have

$$\|\mathbf{u}\|_1 = \sum_{j=1}^{n} |u_j| \le \sum_{j=1}^{n} \max_{1\le k\le n} |u_k| \le n \max_{1\le k\le n} |u_k| = n\|\mathbf{u}\|_\infty \ ,$$

(where we used the obvious fact that $|u_j| \le \max\limits_{1\le k\le n} |u_k|$ for any $j = 1, \ldots, n$), and

$$\|\mathbf{u}\|_\infty = \max_{1\le k\le n} |u_k| \le \sum_{j=1}^{n} |u_j| = \|\mathbf{u}\|_1 \ ,$$

hence

$$\|\mathbf{u}\|_\infty \le \|\mathbf{u}\|_1 \le n\|\mathbf{u}\|_\infty \ ,$$

which proves our claim (for the choice of constants $A = 1$, $B = n$).

(a) Prove that the norms $\| \ \|_2$ and $\| \ \|_\infty$ are equivalent.

(b) Prove that the norms $\| \ \|_1$ and $\| \ \|_2$ are equivalent.
Hint: From the fact that $\| \ \|_1$ and $\| \ \|_\infty$ are equivalent, and the fact that $\| \ \|_2$ and $\| \ \|_\infty$ are equivalent (proved in part (a)), you can solve part (b) without any additional calculations.

Problem 2. Many theorems that hold in finite-dimensional spaces are not true in infinite-dimensional spaces. One can think of the real infinite-dimensional space $\mathbb{R}^\infty$ as the space of infinite sequences: $\mathbf{u} = (u_1, u_2, u_3, \ldots)$, where u_j are real numbers ($j \in \mathbb{N} := \{1, 2, 3, \ldots\}$). In this space we can define the norms $\| \ \|_1$, $\| \ \|_2$, and $\| \ \|_\infty$ as usual:

$$\|\mathbf{u}\|_1 := \sum_{j\in\mathbb{N}} |u_j| \ , \qquad \|\mathbf{u}\|_2 := \left(\sum_{j\in\mathbb{N}} |u_j|^2 \right)^{1/2} \ , \qquad \|\mathbf{u}\|_\infty := \sup_{j\in\mathbb{N}} |u_j| \ .$$

Here $\sup_{j\in\mathbb{N}} a_j$ (the "supremum") is the smallest number a such that $a_j \le a$ for all $j \in \mathbb{N}$. The supremum over a finite set of real numbers is the same as the maximum over this set. For an infinite set, however, the set may not have a maximum, but it always has a supremum (which may be finite or infinite); for example, the set $\{5-1, 5-\frac{1}{2}, 5-\frac{1}{3}, 5-\frac{1}{4}, \ldots, 5-\frac{1}{k}, \ldots\}$ has a supremum (equal to 5), but does not have a maximum (because none of the elements of the set is *equal* to 5).

(a) Give an explicit example of a sequence $\mathbf{u}$ such that $\|\mathbf{u}\|_\infty < \infty$, but $\|\mathbf{u}\|_1$ is infinite.
Hint: How about $\mathbf{u} = (1, 1, 1, \ldots)$?

(b) Give an explicit example of a sequence $\mathbf{u}$ such that $\|\mathbf{u}\|_\infty < \infty$, but $\|\mathbf{u}\|_2$ is infinite.

(c) Give an explicit example of a sequence $\mathbf{u}$ such that $\|\mathbf{u}\|_2 < \infty$, but $\|\mathbf{u}\|_1$ is infinite.
Hint: Think how you can use the following facts:

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} , \qquad \sum_{j=1}^{\infty} \frac{1}{j} = \infty .$$

Problem 3. In this problem, a "polynomial" means a polynomial of a real variable with real coefficients (so that both x and $P(x)$ are real numbers). As discussed in class, the polynomials of order no higher than n form a linear space with respect to the addition of polynomials and multiplication of a polynomial by a real number as follows: if P and Q are polynomials of degree $\le n$ and $\alpha \in \mathbb{R}$, then the polynomials $P + Q$ and αP are defined as

$$(P+Q)(x) := P(x) + Q(x) , \qquad (\alpha P)(x) := \alpha\, P(x) .$$

Let $V_n(a, b; w(x))$ stand for the linear space of polynomials defined on the interval with left end a and right end b (at each end, the interval can be open or closed; a and b can be finite or infinite) of degree no greater than n endowed with the inner product

$$\langle P, Q\rangle = \int_a^b P(x)\, Q(x)\, w(x)\, \mathrm{d}x .$$

Samer defined a family of polynomials which he denoted (very modestly!) by S_0, S_1, S_2, ... These polynomials satisfy the following conditions:

(i) the polynomial S_k is of degree k;

(ii) the coefficient of x^k in S_k is equal to 1 (such polynomials are called *monic*);

(iii) the polynomials S_0, S_1, S_2, ..., S_n form an orthogonal basis in the space of polynomials $V_n(0, \infty; w(x) = \mathrm{e}^{-x})$.

In the solution of this problem the following identity will be handy:

$$\int_0^\infty x^k \, \mathrm{e}^{-x} \, \mathrm{d}x = k!$$

(where, by definition, $0! = 1$).

(a) Clearly, $S_0(x) = 1$ for each $x \in [0, \infty)$. Find the only monic polynomial S_1 of degree 1 that is orthogonal to S_0 (i.e., such that $\langle S_1, S_0 \rangle = 0$).

(b) Find the only monic quadratic polynomial S_2 that is orthogonal to both S_0 and S_1.

(c) Show that the polynomial $P(x) = x^2 + 3$ can be represented as a linear combination of the polynomials S_0, S_1 and S_2 as follows: $P = S_2 + 4S_1 + 5S_0$.

(d) Show by direct integration that $\langle S_0, S_0 \rangle = 1$, $\langle S_1, S_1 \rangle = 1$, $\langle S_2, S_2 \rangle = 4$.

(e) Find the orthogonal projection, $\mathrm{proj}_{S_0+2S_1} P$, of the polynomial $P(x) = x^2 + 3$ onto the " straight line"

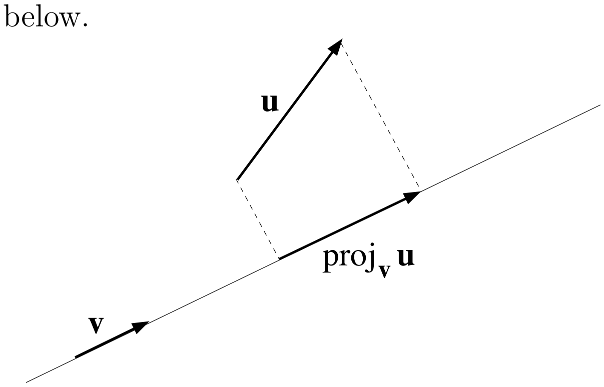
$$\ell := \{t(S_0 + 2S_1) \mid t \in \mathbb{R}\}$$

in the 3-dimensional inner product linear space $V_2(0, \infty; \mathrm{e}^{-x})$. If you have solved part (c), then finding this orthogonal projection should be easy.

Hint: If **u** and **v** are vectors in the inner product linear space V, then the orthogonal projection of the vector **u** onto the straight line in the direction of **v** is the vector

$$\mathrm{proj}_{\mathbf{v}} \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \, \mathbf{v}$$

– see the picture below.



(f) Finally, let $\widetilde{S}_k := \mu_k S_k$, where $\mu_k > 0$ is a constant (depending on k) such that the norm,

$$\left\| \widetilde{S}_k \right\| := \sqrt{\left\langle \widetilde{S}_k, \widetilde{S}_k \right\rangle} \, ,$$

of the polynomial $\widetilde{S}_k$ is 1. Find the explicit expressions for $\widetilde{S}_0(x)$, $\widetilde{S}_1(x)$, and $\widetilde{S}_2(x)$.