MATH 3423 Homework 1 Due Thu, 08/29/19

Problem 1. Consider the action functional

$$I[q] = \int_{t_1}^{t_2} (q \sin\dot{q} + t\,\dot{q})\,\mathrm{d}t\ , \tag{1}$$

where $q(t)$ is a function of the time t, and $\dot{q}(t)$ is its derivative: $\dot{q}(t) = \frac{\mathrm{d}}{\mathrm{d}t}q(t)$.

(a) Write down the Lagrangian $L(q, \dot{q}, t)$ for the action functional $I[t]$ given by (1).

$$L(q, \dot{q}, t) = q \sin\dot{q} + t\,\dot{q}\ .$$

(b) Write down the derivative $\dfrac{\partial L}{\partial q}$.

(c) Write down the derivative $\dfrac{\partial L}{\partial \dot{q}}$.

(d) Treating $q(t)$ and $\dot{q}(t)$ as functions of t, write down $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \dot{q}}\right)$; expand your result fully, do not leave any parentheses.

Problem 2. Recall that for action

$$I[q] = \int_{t_i}^{t_f} L(q(t), \dot{q}(t), t)\,\mathrm{d}t$$

(note that the Lagrangian is allowed to depend explicitly on t), the Euler-Lagrange equations are given by

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = 0\ .$$

(a) Derive the Euler-Lagrange equation for the action functional

$$I[q] = \int_{t_1}^{t_2} t\sqrt{1 - \dot{q}^2}\,\mathrm{d}t\ .$$

Please write down your calculations in detail.

(b) Derive the Euler-Lagrange equation for the action functional

$$I[q] = \int_{t_1}^{t_2} \left(t\dot{q}^2 - q\dot{q} + q\right)\,\mathrm{d}t\ .$$

Please write down your calculations in detail.

Problem 3. Consider the action functional

$$I[q] = \int_{t_1}^{t_2} \dot{q} \left(1 + t^2 \dot{q} \right) \, \mathrm{d}t \ . \tag{2}$$

(a) Derive the Euler-Lagrange equation for the action functional $I[q]$ given by (2).

(b) Look at the Euler-Lagrange equation written in part (a). Recall that if $\frac{\mathrm{d}}{\mathrm{d}t} g(t) = 0$, then $g(t) = \text{const.}$ Use this to reduce the Euler-Lagrange from part (a) to a first-order ODE; it will be a simple separable ODE.

(c) Find the general solution of the separable ODE derived in part (b) (I want to see your calculations). Show that the general solution you found can be written in the form

$$q(t) = \frac{C_1}{t} + C_2 \ , \tag{3}$$

where C_1 and C_2 are arbitrary constants.

(d) Impose the boundary conditions

$$q(1) = 5 \ , \qquad q(2) = 3$$

on the solution $q(t)$ on the interval $t \in [1, 2]$ to find the particular solution of the Euler-Lagrange equation for the action functional (2) that you have derived in part (b). In other words, find the values of the constants C_1 and C_2 in (3).

(e) Denote the particular solution derived in part (d) by $\bar{q}(t)$. Find the numerical value of the action integral

$$I[\bar{q}] = \int_1^2 \dot{\bar{q}}(t) \left[1 + t^2 \dot{\bar{q}}(t) \right] \, \mathrm{d}t \ .$$

for the particular choice of function $\bar{q}(t)$, and for $t_1 = 1$, $t_2 = 2$.

Problem 4. Consider a curve in the plane defined by the parametric equations

$$x = q_1(t) \ , \qquad y = q_2(t) \ , \tag{4}$$

where $q_1(t)$ and $q_2(t)$ are some functions of t.

(a) The functional $I[q_1, q_2] = \int_a^b \sqrt{\dot{q}_1^2 + \dot{q}_2^2} \, \mathrm{d}t$ describes the length of the curve (4) between the points $(q_1(a), q_2(a))$ and $(q_1(b), q_2(b))$. Find the functions $q_1(t)$ and $q_2(t)$ that extremize this functional. (In fact, one can show (and it is obvious geometrically) that these functions minimize the functional, but you do not have to do this here.)

(b) Your result from part (a) has a very simple geometric meaning – what is it?