Additional problem 1:

(a)
$$\begin{cases} \Delta u = 0, \quad x \in [0,a],\ y \in [0,b] \\ u(0,y) = 0,\ u(a,y) = 0 \\ u(x,0) = 0,\ u(x,b) = 5 \sin \frac{7\pi x}{a}. \end{cases}$$

The solution has the form

$$u(x,y) = \sum_{n=1}^{\infty} Y_n(y) \sin \frac{n\pi x}{a}, \qquad (*)$$

where the functions $Y_n(x)$ satisfy

$$Y_n'' - \left(\frac{n\pi}{a}\right)^2 Y_n = 0. \qquad (\#)$$

Then the function $u(x,y)$ defined by $(*)$ is a solution of the PDE

$$\Delta u = 0$$

and satisfies the BCs

$$u(0,y) = 0, \quad u(a,y) = 0.$$

The general solution of the equation $(\#)$ for $Y_n(y)$ is

$$Y_n(y) = A_n \cosh \frac{n\pi y}{a} + B_n \sinh \frac{n\pi y}{a}.$$

The constants A_n and B_n come from the remaining two ICs; using that $\cosh 0 = 1$, $\sinh 0 = 0$, we obtain

$$\begin{cases} u(x,0) = \sum_{n=1}^{\infty} \left(A_n \cdot 1 + B_n \cdot 0\right) \sin \frac{n\pi x}{a} = 0 \\ u(x,b) = \sum_{n=1}^{\infty} \left(A_n \cosh \frac{n\pi b}{a} + B_n \sinh \frac{n\pi b}{a}\right) \sin \frac{n\pi x}{a} \\ \qquad = 5 \sin \frac{7\pi x}{a}. \end{cases}$$

From the first condition we obtain that all A_n's are 0, and the second condition then becomes

$$\sum_{n=1}^{\infty} B_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a} = 5 \sin \frac{7\pi x}{a}$$

$$\Longrightarrow B_7 = \frac{5}{\sinh \frac{7\pi b}{a}}, \quad B_n = 0 \text{ for } n \neq 7.$$

Putting all this together:

$$\boxed{u(x,y) = \frac{5}{\sinh \frac{7\pi b}{a}} \sinh \frac{7\pi y}{a} \sin \frac{7\pi x}{a}}.$$

(b) Using the hint in the statement of the problem, we impose the condition

$$Y_n(b) = 0$$

(coming from $u(x,b) = 0$) on $Y_n(y)$:

$$Y_n(b) = C_n \cosh\frac{n\pi b}{a} + D_n \sinh\frac{n\pi b}{a} = 0,$$

which implies that

$$C_n = -D_n \frac{\sinh\frac{n\pi b}{a}}{\cosh\frac{n\pi b}{a}} = -D_n \tanh\frac{n\pi b}{a}$$

$$\Rightarrow Y_n(b) = \underbrace{-D_n \tanh\frac{n\pi b}{a}}_{C_n} \cosh\frac{n\pi y}{a} + D_n \sinh\frac{n\pi y}{a}$$

$$= -\frac{D_n}{\cosh\frac{n\pi b}{a}}\left(\sinh\frac{n\pi b}{a}\cosh\frac{n\pi y}{a} - \cosh\frac{n\pi b}{a}\sinh\frac{n\pi y}{a}\right)$$

$$= E_n \sinh\frac{n\pi(b-y)}{a},$$

where we have set $E_n := -\dfrac{D_n}{\cosh\frac{n\pi b}{a}}$ to be the relabeled arbitrary constant.

Now we have

$$u(x,y) = \sum_{n=1}^{\infty} E_n \sinh\frac{n\pi(b-y)}{a} \sin\frac{n\pi x}{a},$$

and imposing the remaining BCs yields

$$u(x,0) = \sum_{n=1}^{\infty} E_n \sinh\frac{n\pi b}{a}\sin\frac{n\pi x}{a} = \sin\frac{3\pi x}{a}$$

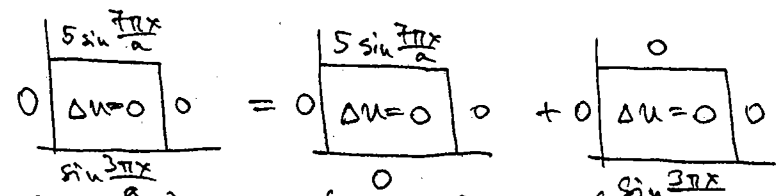
$\Rightarrow$ all E_n except E_3 are 0, and

$$E_3 = \frac{1}{\sinh\frac{3\pi b}{a}}.$$

This yields

$$\boxed{u(x,y) = \frac{1}{\sinh\frac{3\pi b}{a}} \sinh\frac{3\pi(b-y)}{a} \sin\frac{3\pi x}{a}}$$

ⓒ We have



(part (c)) = (part (a)) + (part (b)),

so the solution of the problem in part (c) is

$$\boxed{u(x,y) = \frac{5}{\sinh\frac{7\pi b}{a}} \sinh\frac{7\pi y}{a} \sin\frac{7\pi x}{a} + \frac{1}{\sinh\frac{3\pi b}{a}} \sinh\frac{3\pi(b-y)}{a} \sin\frac{3\pi x}{a}.}$$