MATH 3113 – Homework assigned on 11/25/13

Problem 1. A *vector space* (or *linear space*) V is a set of elements called *vectors* with two operations – *addition* of two vectors (which takes two vectors, $\mathbf{u} \in V$ and $\mathbf{v} \in V$, and gives their sum, the vector $\mathbf{u}+\mathbf{v} \in V$), and multiplication of a number and a vector (which takes the number $\alpha \in \mathbb{R}$ and the vector $\mathbf{u} \in V$, and gives the vector $\alpha\mathbf{u} \in V$). As we learned in class, functions can be considered as vectors if we define the operations addition of two functions and multiplication of a number and a function – as you know from middle school (although you did not put it in such a formal language), for given functions f and g and a number α, the functions $f+g$ and αf are defined by

$$(f+g)(x) := f(x) + g(x) \ , \qquad (\alpha f)(x) := \alpha\, f(x) \ .$$

One can endow a vector space with additional structures, like a norm and an inner product. An *inner product* is an operation that takes two vectors, $\mathbf{u} \in V$ and $\mathbf{v} \in V$, and gives a number $\langle \mathbf{u}, \mathbf{v} \rangle \in \mathbb{R}$; this operation must satisfy the properties

(a) symmetry: $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ for all $\mathbf{u}\,, \mathbf{v} \in V$;

(b) linearity: $\langle \alpha\mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ for all $\alpha \in \mathbb{R}$, and all $\mathbf{u}\,, \mathbf{v}\,, \mathbf{w} \in V$;

(c) $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ for all $\mathbf{u} \in V$; moreover, $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ implies that $\mathbf{u} = \mathbf{0}$ (where $\mathbf{0} \in V$ is the zero vector which is the only vector with the property that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for any $\mathbf{v} \in V$).

A vector space endowed with an inner product is called an *inner product vector space*. In geometry the inner product is often called *dot product* and denoted by $\mathbf{u} \cdot \mathbf{v}$. Two vectors, $\mathbf{u}$ and $\mathbf{v}$, are said to be *orthogonal* if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

The standard high-school example of inner product between two vectors from $\mathbb{R}^n$, $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, is defined as

$$\langle \mathbf{u}, \mathbf{v} \rangle := \sum_{i=1}^{n} u_i v_i \ .$$

This, however, is not the only possible inner product in $\mathbb{R}^n$: any $n \times n$ matrix $\mathbf{Q} = (q_{ij})$ with the property that $\mathbf{u}^{\mathrm{T}}\mathbf{Q}\mathbf{u} > 0$ for any $\mathbf{u} \neq \mathbf{0}$ defines an inner product $\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{Q}}$ by

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{Q}} := \sum_{i=1}^{n} \sum_{j=1}^{n} u_i q_{ij} v_j \ . \tag{1}$$

Function spaces (i.e., vector spaces where the elements are functions) can also be endowed with inner products. Similarly to (1), one can define inner product of two functions f and g, both defined on the interval $[a, b]$, as

$$\langle f, g \rangle_w := \int_a^b f(x)\, g(x)\, w(x)\, \mathrm{d}x \ , \tag{2}$$

where $w(x)$ is a function defined on $[a, b]$ such that $\displaystyle\int_a^b w(x)\, \mathrm{d}x$ exists, $w(x) \geq 0$ for all $x \in [a, b]$, and $w(x)$ is allowed to be zero only at isolated points in $[a, b]$.

The concept of a *basis* of a vector space V is the same as in elementary geometry – a basis is a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ such that every vector $\mathbf{u} \in V$ can be written in the form $\mathbf{u} = \sum_{i=1}^{n} u_i \mathbf{v}_i$ in a unique way; the numbers u_i are called the *components* of the vector $\mathbf{u}$ in the basis $\mathbf{v}_i$. The number of vectors in a basis is called the *dimension* of the vector space V (one can prove that the number of vectors in every basis is the same, so that the definition of dimension makes sense).

If there is an inner product defined in V, then one can choose the basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ such that $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ if $i \neq j$ – such a basis is called *orthogonal.* If a basis is orthogonal, then the components u_j of a vector $\mathbf{u}$ can be found very easily: take the inner product of

$$\mathbf{u} = u_1 \mathbf{v}_1 + u_2 \mathbf{v}_2 + \cdots + u_n \mathbf{v}_n$$

with each of the vectors in the basis to obtain $\langle \mathbf{u}, \mathbf{v} \rangle = u_j \langle \mathbf{v}_j, \mathbf{v}_j \rangle$ (because $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for any $i \neq j$), which implies that $u_j = \dfrac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}_j, \mathbf{v}_j \rangle}$.

In this problem you will construct an orthogonal basis q_0, q_1, q_2, q_3 in the vector space of polynomials of degree no more than 3, defined on the interval $[0, 1]$ and endowed with the inner product

$$\langle f, g \rangle := \int_0^1 f(x)\, g(x)\, \mathrm{d}x \; ; \tag{3}$$

let us denote this vector space by $\mathcal{P}_{3,[0,1]}$. The space of polynomials of degree no higher than n is $(n+1)$-dimensional because such a polynomial has $(n+1)$ coefficients:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

If one chooses the polynomials 1, x, x^2, …, x^n as the basis of this space, then the components of $p(x)$ are the numbers a_0, a_1, …, a_n.

The orthogonal basis of the vector space $\mathcal{P}_{3,[0,1]}$ that you will construct below will satisfy two additional conditions: the degree of the polynomial $q_i(x)$ will be i, and each polynomial q_i will be *monic*, i.e., the the coefficient in front of the highest power of x is equal to 1; these two conditions combined imply that $q_0(x) = 1$, $q_1(x) = x + \cdots$, $q_2(x) = x^2 + \cdots$, $q_3(x) = x^3 + \cdots$.

(a) Construct the monic polynomial $q_1(x) = x + \alpha$ such that q_1 is orthogonal to q_0, i.e., choose the coefficient α in such a way that $\langle q_1, q_0 \rangle = 0$.

(b) Construct the monic polynomial $q_2(x) = x^2 + \beta x + \gamma$ such that q_2 is orthogonal to q_0 and q_1. The conditions $\langle q_2, q_0 \rangle = 0$ and $\langle q_2, q_1 \rangle = 0$ give you two equations for the two unknown coefficients β and γ.

(c) Construct the monic polynomial $q_3(x) = x^3 + \mu x^2 + \nu x + \rho$ orthogonal to q_0, q_1, and q_2. Since the calculations are getting tedious (and any error will give you a completely wrong final result), here is the answer:

$$q_3(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20} \; . \tag{4}$$

You only need to write down the equations for the unknown coefficients μ, ν, and ρ, and check that the coefficients in (4) satisfy them.

(d) Show that the polynomial $p(x) = x^3$ can be written in the basis q_i as

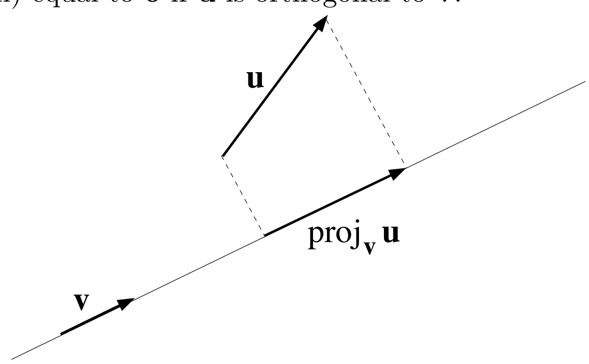
$$p = q_3 + \frac{3}{2}q_2 + \frac{9}{10}q_1 + \frac{1}{4}q_0 \; . \tag{5}$$

I don't want you only to check that these coefficients in (5) are indeed the correct ones, but to approach the problem as if you did not know the coefficients in (5) and wanted to find them. In other words, show me what you would do – don't finish the calculations, just write clearly the expression/procedure that you would use to find the coefficients.

(e) Recall from elementary geometry that a straight line in the direction of the non-zero $\mathbf{v} \in \mathbb{R}^n$ through the origin of $\mathbb{R}^n$ can be written as $\ell = \{t\mathbf{v} \,|\, t \in \mathbb{R}\}$ (the condition $\mathbf{v} \neq \mathbf{0}$ is imposed simply because the zero vector does not have a direction). The *orthogonal projection* of a vector $\mathbf{u} \in \mathbb{R}^n$ onto the straight line ℓ in the direction of $\mathbf{v}$ is the vector

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\, \mathbf{v} \tag{6}$$

– see the picture below. You can check (but don't need to write anything here) that the expression (6) indeed defines a vector $\text{proj}_{\mathbf{v}}\mathbf{u}$ that satisfies the following reasonable properties justifying the words "orthogonal projection": (i) proportional to $\mathbf{v}$, (ii) such that if $\mathbf{u} = \alpha\mathbf{v}$, then $\text{proj}_{\mathbf{v}}\mathbf{u} = \mathbf{u}$, (iii) equal to $\mathbf{0}$ if $\mathbf{u}$ is orthogonal to $\mathbf{v}$.



Similarly, one can define a straight line in the vector space $\mathcal{P}_{3,[0,1]}$. Let

$$\ell = \{t(q_0 + 2q_1) \,|\, t \in \mathbb{R}\} \tag{7}$$

be the straight line containing the vector $q_0 + 2q_1 \in \mathcal{P}_{3,[0,1]}$. Find the orthogonal projection, $\text{proj}_{q_0+2q_1}p$, of the vector $p \in \mathcal{P}_{3,[0,1]}$ defined by $p(x) = x^3$ onto the straight line ℓ (7), by using the coefficients of p in the basis q_i given in (5).

Remark: You may need to use some of the following (straightforward, but tedious to obtain by hand) results:

$$\langle q_0, q_0 \rangle = 1 \; , \quad \langle q_1, q_1 \rangle = \frac{1}{12} \; , \quad \langle q_2, q_2 \rangle = \frac{1}{180} \; , \quad \langle q_3, q_3 \rangle = \frac{1}{2800} \; .$$