MATH 3113 – Homework assigned on 10/25/13

Problem 1. Let $u(t)$ be the Heaviside function (see pages 446-447 of the book), and a is an arbitrary positive number.

(a) Sketch the graph of the function $u(t-3)f(t-3)$ in the particular case $f(t) = t$.

(b) Directly from the definition of Laplace transform, prove that, for an arbitrary function $f : [0, \infty) \to \mathbb{R}$ whose Laplace transform $F(s)$ exists,

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}\,F(s)\ .$$

(c) Use the formula from part (b) to find the inverse Laplace transform of the function $F(s) = \dfrac{e^{-s} - e^{-3s}}{s^2}$, and sketch the graph of $\mathcal{L}^{-1}\{F\}(t)$.

Problem 2. Let $f(t)$ be a function defined for $t \geq 0$ that is periodic of period $p > 0$, i.e., that satisfies

$$f(t+p) = f(t) \qquad \text{for every } t \geq 0\ .$$

(a) Prove that, for any non-negative integer n,

$$\int_{np}^{(n+1)p} e^{-st}\,f(t)\,\mathrm{d}t = e^{-nps}\int_0^p e^{-st}\,f(t)\,\mathrm{d}t\ .$$

(b) Explain why the Laplace transform $F(s)$ of $f(t)$ can be written in the form

$$F(s) = \sum_{n=0}^{\infty}\int_{np}^{(n+1)p} e^{-st}\,f(t)\,\mathrm{d}t\ .$$

(c) Use your results from parts (a) and (b) to show that

$$F(s) = \frac{1}{1-e^{-ps}}\int_0^p e^{-st}\,f(t)\,\mathrm{d}t\ .$$

Hint: $e^{-psn} = (e^{-ps})^n$; for $s > 0$ and $p > 0$ (the case we consider here), $0 < e^{-ps} < 1$.

(d) Apply formula obtained in part (c) to solve Problem 7.1/41 (which is the same as Problem 7.2/34).

Hint: We solved this problem in detail in Lecture 23 on October 14.

Problem 3. In class we defined the delta function $\delta_a(t)$ by the relation

$$\int_{-\infty}^{\infty} \delta_a(t)\, f(t)\, \mathrm{d}t := f(a) \ ,$$

where $f(t)$ is an arbitrary continuous function. (It is customary to even require that $f(t)$ be a function that is infinitely many times differentiable, but we will not worry about this at the level of rigor of this class.) In class we motivated the definition above by considering a function

$$d_{a,\varepsilon}(t) := \begin{cases} \dfrac{1}{\varepsilon} & \text{for } x \in (a, a+\varepsilon) \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

One can think of $\delta_a(t)$ as a formal limit $\lim\limits_{\varepsilon \to 0^+} d_{a,\varepsilon}(t)$, in the sense that

$$\int_{-\infty}^{\infty} \delta_a(t)\, f(t)\, \mathrm{d}t := \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} d_{a,\varepsilon}(t)\, f(t)\, \mathrm{d}t \ .$$

(a) Compute $\displaystyle\int_{-\infty}^{\infty} d_{a,\varepsilon}(t)\, \cos t\, \mathrm{d}t$.

(b) Compute $\displaystyle\lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} d_{a,\varepsilon}(t)\, \cos t\, \mathrm{d}t$. Discuss your findings.

Hint: You may find useful the following facts:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \ , \qquad \lim_{h\to 0} \frac{\sin h}{h} = 1 \ , \qquad \lim_{h \to 0} \frac{1-\cos h}{h} = 0 \ .$$

(c) Let $f(t)$ be an arbitrary continuous function, and let $F(t)$ be its antiderivative. Express $\displaystyle\int_{-\infty}^{\infty} d_{a,\varepsilon}(t)\, f(t)\, \mathrm{d}t$ in terms of F.

(d) Compute $\displaystyle\lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} d_{a,\varepsilon}(t)\, f(t)\, \mathrm{d}t$. Please write specifically what facts you have used. Discuss what your result in the light of the connection between $d_{a,\varepsilon}(t)$ and $\delta_a(t)$, and the definition of δ_a.