Additional problem assigned on 9/26/2017

Additional problem. Consider the function defined implicitly by the relation

\[ x^3 - y^3 = 7. \]  

(1)

(a) Prove that the point \((x^*, y^*) = (2, 1)\) belongs to the curve described by the equation (1).

(b) Use implicit differentiation to find the derivative \(\frac{dy}{dx}\) as a function of \(x\) and \(y\).

(c) Compute the value of \(\frac{dy}{dx}\bigg|_{(2,1)}\).

(d) Write down the equation of the tangent line to the curve described by the equation (1) at the point \((2, 1)\). Figure 1 shows the curve and the tangent line to it at the point \((2, 1)\).

(e) Use implicit differentiation again to show that the second derivative \(\frac{d^2y}{dx^2}\) as a function of \(x\) and \(y\) is given by

\[
\frac{d^2y}{dx^2} = \frac{2x}{y^2} \left( 1 - \frac{x^3}{y^3} \right) = \frac{2x}{y^2} \frac{y^3 - x^3}{y^3} = -\frac{14x}{y^5}.
\]

What did you use at the last step in this chain of equalities?

(f) Find the value \(\frac{d^2y}{dx^2}\bigg|_{(2,1)}\) of the second derivative at the point \((2, 1)\).

Figure 1: The curve described by equation (1) and the tangent line to it at the point \((2, 1)\).