MATH 3413 **Problems assigned on 4/22/14**

Problem 1. Consider the problem about the stationary temperature distribution in the rectangle $x \in [0, a]$, $y \in [0, b]$ (which can be symbolically written as $(x, y) \in [0, a] \times [0, b]$) if there are no sources of heat in the rectangle (hence the temperature $u(x, y)$ satisfies Laplace's equation $\Delta u = 0$), and the temperature at the sides of the rectangle is maintained as follows:

$$\begin{aligned} &u(0, y) = 0 \ , \qquad u(a, y) = 0 \qquad \text{for } y \in [0, b] \\ &u(x, 0) = \sin \frac{3\pi x}{a} \ , \qquad u(x, b) = 5 \sin \frac{7\pi x}{a} \qquad \text{for } x \in [0, a] \ . \end{aligned}$$

(a) Solve the boundary value problem

$$\begin{aligned} &\Delta u = 0 \ , \qquad (x, y) \in [0, a] \times [0, b] \\ &u(0, y) = 0 \ , \qquad u(a, y) = 0 \qquad \text{for } y \in [0, b] \\ &u(x, 0) = 0 \ , \qquad u(x, b) = 5 \sin \frac{7\pi x}{a} \qquad \text{for } x \in [0, a] \ . \end{aligned}$$

(b) Solve the boundary value problem

$$\begin{aligned} &\Delta u = 0 \ , \qquad (x, y) \in [0, a] \times [0, b] \\ &u(0, y) = 0 \ , \qquad u(a, y) = 0 \qquad \text{for } y \in [0, b] \\ &u(x, 0) = \sin \frac{3\pi x}{a} \ , \qquad u(x, b) = 0 \qquad \text{for } x \in [0, a] \ . \end{aligned}$$

Hint: Let $Y_n(y)$ stands for the functions in the expansion

$$u(x, y) = \sum_{n=1}^{\infty} Y_n(y) \, X_n(x) \ ,$$

where because of the homogeneous boundary conditions at $x = 0$ and $x = a$ the functions $X_n(x)$ are given by $X_n(x) = \sin \frac{n\pi x}{a}$. Then the general solution of the ODE for $Y_n(y)$ is

$$Y_n(y) = C_n \cosh \frac{n\pi y}{a} + D_n \sinh \frac{n\pi y}{a} \ .$$

Show that the homogeneous boundary condition at $y = b$ implies that

$$\begin{aligned} Y_n(y) &= E_n \left(\sinh \frac{n\pi b}{a} \cosh \frac{n\pi y}{a} - \cosh \frac{n\pi b}{a} \sinh \frac{n\pi y}{a} \right) \\ &= E_n \sinh \frac{n\pi (b - y)}{a} \end{aligned}$$

(where E_n are constants arbitrary at the moment); here we have used the fact that hyperbolic sine satisfies

$$\sinh(\alpha \pm \beta) = \sinh \alpha \cosh \beta \pm \cosh \alpha \sinh \beta \ .$$

Now impose the remaining boundary condition to find the constants E_n (of which only one will be non-zero).

(c) Since the equation is linear and homogeneous (i.e., with a zero right-hand side), the principle of superposition holds similarly to the case of ordinary differential equations. Using this fact, write down the solution of the boundary value problem

$$\begin{aligned} &\Delta u = 0 \ , \qquad (x,y) \in [0,a] \times [0,b] \\ &u(0,y) = 0 \ , \qquad u(a,y) = 0 \qquad \text{for } y \in [0,b] \\ &u(x,0) = \sin\frac{3\pi x}{a} \ , \qquad u(x,b) = 5\sin\frac{7\pi x}{a} \qquad \text{for } x \in [0,a] \ . \end{aligned}$$

Problem 2. Solve the boundary value problem

$$\begin{aligned} &\Delta u(x,y) = 0 \\ &u(0,y) = 0 \ , \qquad u(a,y) = 0 \qquad \text{for } y \in [0,\infty) \\ &u(x,0) = \sin\frac{3\pi x}{a} \qquad \text{for } x \in [0,a] \end{aligned}$$

in the semi-infinite strip $x \in [0,a]$, $y \in [0,\infty)$. From physical point of view it is quite clear that we have to also impose the condition $\lim_{y\to\infty} u(x,y) = 0$.

Hint: When you are trying to find the functions $Y_n(y)$, it will be more convenient to write them as superposition of exponents rather than as superposition of hyperbolic functions (because $e^{-(\text{positive constant})\,y}$ tends to 0 while $e^{(\text{positive constant})\,y}$ tends to infinity as $y \to \infty$). *Reading Example 2 from Section 9.7 of the book (on pages 648, 649) will be VERY useful!*

Problem 3. Solve the boundary value problem

$$\begin{aligned} &\Delta u(x,y) = 0 \ , \qquad (x,y) \in [0,a] \times [0,b] \\ &u_x(0,y) = 0 \ , \qquad u_x(a,y) = 0 \qquad \text{for } y \in [0,\infty) \\ &u(x,0) = 0 \ , \qquad u(x,b) = 7 + \cos\frac{3\pi x}{a} \qquad \text{for } x \in [0,a] \ . \end{aligned}$$

Note that the boundary conditions on the walls $x = 0$ and $x = a$ are of Neumann type, which tells you that you have to look for an expansion of $u(x,y)$ in the form

$$u(x,y) = Y_0(y)\,X_0(x) + \sum_{n=1}^{\infty} Y_n(y)\,X_n(y) \ ,$$

where $X_0(x) = 1$, $X_n(x) = \cos\frac{n\pi x}{a}$ $(n = 1, 2, \ldots)$. The functions $Y_n(y)$ for $n = 1, 2, \ldots$ are the same as in the Hint to Problem 1(b) above. How about the function $Y_0(y)$?