MATH 3413 **Problems assigned on 4/15/14**

Problem 1. In this problem you can use without deriving it that the solution of the wave equation $u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0$, $x \in [0, L]$, $t \geq 0$, with homogeneous Dirichlet BCs $u(0,t) = 0$, $u(L,t) = 0$ has the form

$$u(x,t) \quad = \quad \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{L} \ .$$

The functions $T_n(t)$ satisfy the ODEs

$$T_n''(t) + \left(\frac{n\pi c}{L}\right)^2 T_n(t) = 0 \ , \qquad t \geq 0 \ ,$$

so their general form is

$$T_n(t) = A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \ .$$

The constants A_n and B_n can be found from the initial conditions, $u(x,0) = g(x)$ (initial position of the spring) and $u_t(x,0) = h(x)$ (initial velocity of the spring).

Solve the BVP

$$\begin{aligned} & u_{tt} - 9u_{xx} = 0 \ , \qquad x \in [0, \pi] \ , \quad t \geq 0 \ , \\ & u(0,t) = 0 \ , \quad u(\pi, t) = 0 \ , \\ & u(x,0) = 4 \sin 2x \ , \qquad u_t(x,0) = 15 \sin 5x \ . \end{aligned}$$

Problem 2. As you know, the solution of the wave equation $u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0$ for $x \in \mathbb{R}$ can always be represented in the form

$$u(x,t) = \phi(x - ct) + \psi(x + ct) \ ,$$

for some functions $\phi(y)$ and $\psi(y)$ of one variable (this is proved in the Food for Thought Problem below). This result holds for *any* solution of the wave equation, even if the spatial variable x belongs only to a finite-length interval, e.g., $x \in [0, L]$ like in Problem 1. Rewrite your result for the function $u(x,t)$ from Problem 1 in the form $u(x,t) = \phi(x - ct) + \psi(x + ct)$. Write explicitly what the functions $\phi(y)$ and $\psi(y)$ are in this case.

Hint: You have to use some trigonometric identities, which may be easily derived from the following facts:

$$\begin{aligned} \sin(\alpha + \beta) &= \sin\alpha \cos\beta + \cos\alpha \sin\beta \ , \\ \cos(\alpha + \beta) &= \cos\alpha \cos\beta - \sin\alpha \sin\beta \ . \end{aligned}$$

Problem 3. Consider the following problem for the wave equation with air resistance term, with homogeneous Dirichlet BCs on the spatial interval $x \in [0, \pi]$:

$$\begin{aligned}
&u_{xx} - 10u_t - u_{tt} = 0 \ , \qquad x \in [0,\pi] \ , \quad t \geq 0 \ , \\
&u(0,t) = 0 \ , \quad u(\pi,t) = 0 \ , \qquad t \geq 0 \ , \\
&u(x,0) = -8\sin 3x + 12\sin 13x \ , \quad u_t(x,0) = 0 \ , \qquad x \in [0,\pi] \ .
\end{aligned}$$

Physically, this problem corresponds to a spring vibrating in air with resistance proportional to the velocity (i.e., to the time derivative $u_t(x,t)$). The coefficient multiplying $u_t(x,t)$ is proportional to the air resistance coefficient.

Because of the homogeneous Dirichlet BCs, it is clear that we should look for an expansion of the unknown function $u(x,t)$ of the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{L} = \sum_{n=1}^{\infty} T_n(t) \sin nx \tag{1}$$

(here $L = \pi$ is the length of the string).

(a) Plug the expansion (1) in the PDE to show that the unknown functions $T_n(t)$ must satisfy the ODEs

$$T_n''(t) + 10\, T_n'(t) + n^2 T_n(t) = 0 \ . \tag{2}$$

(b) The initial conditions for the functions $T_n(t)$ come from the initial conditions for $u(x,t)$. Plug the expansion (1) into the initial conditions for $u(x,t)$ to show that $T_n(0)$ and $T_n'(0)$ are zero for all n except for $n = 3$ and $n = 13$. What are the initial conditions $T_3(0)$ and $T_3'(0)$ for $T_3(t)$, and the initial conditions $T_{13}(0)$ and $T_{13}'(0)$ for $T_{13}(t)$?

(c) Since the ODEs (2) are homogeneous (i.e., have zero right-hand sides), the solutions for all functions $T_n(t)$ with n not equal to 3 or 13 will be identically equal to zero.

Solve the initial-value problem for the function $T_3(t)$.

(d) Solve the initial-value problem for the function $T_{13}(t)$.

(e) Write down the solution,

$$u(x,t) = T_3(t) \sin 3x + T_{13}(t) \sin 13x \ ,$$

with the functions $T_3(t)$ and $T_{13}(t)$ found in parts (c) and (d).

(f) From the physical interpretation of the problem, what would you expect the asymptotic position of the string to be. No calculation is needed here, only a couple of sentences of explanation.

(g) Does the solution found in part (e) behave as you predicted on physical grounds in part (f)?

Food for Thought Problem [Not to be turned in!]

Consider the wave equation for the function $u(x,t)$ of one spatial (x) and one temporal (t) variables:

$$u_{tt}(x,t) - c^2\, u_{xx}(x,t) = 0\ , \qquad x \in \mathbb{R}\,,\ t \in [0,\infty)\ ,$$

where c is the speed of the wave (measured in meters per second).

(a) Let $\sigma = \Sigma(x,t)$ and $\gamma = \Gamma(x,t)$ be new variables defined as

$$\sigma = \Sigma(x,t) := x - ct\,, \quad \gamma = \Gamma(x,t) := x + ct\ .$$

Let $\widetilde{u}(\sigma,\gamma)$ be a function of two variables defined as

$$u(x,t) := \widetilde{u}\big(\Sigma(x,t), \Gamma(x,t)\big) = \widetilde{u}(\sigma,\gamma)|_{\sigma=\Sigma(x,t),\ \gamma=\Gamma(x,t)}$$

Using the standard jargon, $\widetilde{u}(\sigma,\gamma)$ is the function $u(x,t)$ expressed in the new variables σ and γ. Using the chain rule, we can express the partial derivatives of $u(x,y)$ through the partial derivatives of $\widetilde{u}(\sigma,\gamma)$ as follows (subscripts stand for partial derivatives):

$$\begin{aligned}
u_t &= \widetilde{u}_\sigma \Sigma_t + \widetilde{u}_\gamma \Gamma_t = -c\widetilde{u}_\sigma + c\widetilde{u}_\gamma \\
u_{tt} &= \left(-c\widetilde{u}_\sigma + c\widetilde{u}_\gamma\right)_t = -c\left(\widetilde{u}_{\sigma\sigma}\Sigma_t + \widetilde{u}_{\sigma\gamma}\Gamma_t\right) + c\left(\widetilde{u}_{\gamma\sigma}\Sigma_t + \widetilde{u}_{\gamma\gamma}\Gamma_t\right) = c^2\left(\widetilde{u}_{\sigma\sigma} - 2\widetilde{u}_{\sigma\gamma} + \widetilde{u}_{\gamma\gamma}\right) \\
u_x &= \widetilde{u}_\sigma \Sigma_x + \widetilde{u}_\gamma \Gamma_x = \widetilde{u}_\sigma + \widetilde{u}_\gamma \\
u_{xx} &= \left(\widetilde{u}_\sigma + \widetilde{u}_\gamma\right)_x = \widetilde{u}_{\sigma\sigma}\Sigma_x + \widetilde{u}_{\sigma\gamma}\Gamma_x + \widetilde{u}_{\gamma\sigma}\Sigma_x + \widetilde{u}_{\gamma\gamma}\Gamma_x = \widetilde{u}_{\sigma\sigma} + 2\widetilde{u}_{\sigma\gamma} + \widetilde{u}_{\gamma\gamma}\ .
\end{aligned}$$

Plugging all these derivatives in the wave equation, we obtain

$$c^2\left(\widetilde{u}_{\sigma\sigma} - 2\widetilde{u}_{\sigma\gamma} + \widetilde{u}_{\gamma\gamma}\right) = c^2\left(\widetilde{u}_{\sigma\sigma} + 2\widetilde{u}_{\sigma\gamma} + \widetilde{u}_{\gamma\gamma}\right)\ ,$$

or, after elementary cancellations,

$$\widetilde{u}_{\sigma\gamma}(\sigma,\gamma) = 0\ .$$

The only thing that you have to do in this part of the problem is to show that the general solution of this PDE is

$$\widetilde{u}(\sigma,\gamma) = \phi(\sigma) + \psi(\gamma)\ ,$$

where ϕ and ψ are arbitrary functions of one variable.

(b) Go to the original variables to show that the general solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ is

$$u(x,t) = \phi(x-ct) + \psi(x+ct)\ .$$

(c) Now consider the initial value problem consisting of the wave equation in part (a) and the initial conditions

$$u(x,0) = g(x) \ , \qquad u_t(x,0) = h(x)$$

(where the subscript t stands for differentiation with the respect to t). In physical terms, $u(x,0)$ is the initial position, and $u_t(x,0)$ is the initial velocity. Show that the functions ϕ and ψ from parts (a) and (b) are related to g and h as follows:

$$\begin{aligned} \phi(x) + \psi(x) &= g(x) \\ -\phi'(x) + \psi'(x) &= \frac{1}{c}\, h(x) \ . \end{aligned}$$

(d) One can integrate the second equation from the system in (c) and solve it for the functions f and g, obtaining

$$\begin{aligned} \phi(x) + \psi(x) &= g(x) \\ -\phi(x) + \psi(x) &= \frac{1}{c} \int_0^x h(s)\, \mathrm{d}s + A \ , \end{aligned}$$

where A is an arbitrary constant. Solve this system to show that

$$\begin{aligned} \phi(x) &= \frac{1}{2} g(x) - \frac{1}{2c} \int_0^x h(s)\, \mathrm{d}s - \frac{A}{2} \\ \psi(x) &= \frac{1}{2} g(x) + \frac{1}{2c} \int_0^x h(s)\, \mathrm{d}s + \frac{A}{2} \ . \end{aligned}$$

(e) Use your result in (d) to show that the solution of the initial value problem

$$\begin{aligned} &u_{tt}(x,t) - c^2\, u_{xx}(x,t) = 0 \ , \qquad x \in \mathbb{R} \ , \quad t \in [0,\infty) \\ &u(x,0) = g(x) \ , \qquad u_t(x,0) = h(x) \ , \qquad \text{for } x \in \mathbb{R} \end{aligned}$$

is

$$u(x,t) = \frac{1}{2}\left[g(x-ct) + g(x+ct)\right] + \frac{1}{2c} \int_{x-ct}^{x+ct} h(s)\, \mathrm{d}s \ .$$

Congratulations! You have derived the so-called *d'Alembert formula* for the solution of the wave equation in one spatial dimension on the whole real line!

(f) Use d'Alembert formula to solve the initial-value problem

$$\begin{aligned} &u_t t(x,t) - c^2\, u_{xx}(x,t) = 0 \ , \qquad x \in \mathbb{R} \ , \quad t \in [0,\infty) \\ &u(x,0) = 0 \ , \qquad u_t(x,0) = x\, e^{-x^2} \ , \qquad \text{for } x \in \mathbb{R} \ . \end{aligned}$$