MATH 3413 Problems assigned on 4/10/14

Problem 1. In all parts of the problem below, you can use without deriving the following solutions of the heat equation $u_t(x,t) = \alpha^2 u_{xx}(x,t)$, $x \in [0, L]$, $t \geq 0$, with appropriate boundary conditions; the first expression is for zero temperature at both boundaries (homogeneous Dirichlet BCs), and the second is for zero heat flux at both boundaries (homogeneous Neumann BCs):

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} b_n \exp\left\{-\left(\frac{n\pi\alpha}{L}\right)^2 t\right\} \sin\frac{n\pi x}{L} \ , \\
u(x,t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp\left\{-\left(\frac{n\pi\alpha}{L}\right)^2 t\right\} \cos\frac{n\pi x}{L} \ .
\end{aligned}
$$

(a) Solve the Dirichlet BVP below, find the asymptotic temperature, $u_\infty(x) := \lim_{t\to\infty} u(x,t)$, and explain why the expression you obtained for $u_\infty(x)$ is physically obvious.

$$
\begin{aligned}
&u_t = 9u_{xx} \ , \qquad x \in [0,\pi] \ , \quad t \geq 0 \ , \\
&u(0,t) = 0 \ , \quad u(\pi,t) = 0 \ , \\
&u(x,0) = 4\sin 2x + 7\sin 5x \ .
\end{aligned}
$$

(b) Derive and use a trigonometric relation to solve the following Dirichlet BVP:

$$
\begin{aligned}
&u_t = u_{xx} \ , \qquad x \in [0,\pi] \ , \quad t \geq 0 \ , \\
&u(0,t) = 0 \ , \quad u(\pi,t) = 0 \ , \\
&u(x,0) = 4\sin 4x \cos 2x \ .
\end{aligned}
$$

Hint: By using that

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \ ,$$

one can derive the relation

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right] \ ,$$

which is useful in finding integrals of the form $\displaystyle\int \sin at \cos bt \, dt$.

(c) Solve the Neumann BVP below, find the asymptotic temperature, $u_\infty(x) := \lim_{t\to\infty} u(x,t)$, and explain why the expression you obtained for $u_\infty(x)$ is physically obvious.

$$
\begin{aligned}
&u_t = 9u_{xx} \ , \qquad x \in [0,5] \ , \quad t \geq 0 \ , \\
&u_x(0,t) = 0 \ , \quad u_x(5,t) = 0 \ , \\
&u(x,0) = 7 + 6\cos 2\pi x \ .
\end{aligned}
$$

(d) Solve the Neumann BVP below. You may use the results of Problem 4 of Section 9.3 without deriving them.

$$\begin{aligned} &u_t = 9u_{xx} \ , \qquad x \in [0,2] \ , \quad t \geq 0 \ , \\ &u_x(0,t) = 0 \ , \quad u_x(2,t) = 0 \ , \\ &u(x,0) = f(x) := \begin{cases} x & \text{for } x \in [0,1] \ , \\ 2-x & \text{for } x \in [1,2] \ . \end{cases} \end{aligned}$$

Problem 2. Consider the following BVP with non-homogeneous Dirichlet BCs:

$$\begin{aligned} &u_t = 9u_{xx} \ , \qquad x \in [0,\pi] \ , \quad t \geq 0 \ , \\ &u(0,t) = 0 \ , \quad u(\pi,t) = 5 \ , \\ &u(x,0) = 0 \ . \end{aligned}$$

(a) Set $u(x,t) = \ell(x) + v(x,t)$, where $\ell(x)$ is a linear function of x that satisfies the conditions $\ell(0) = 0$ and $\ell(\pi) = 5$ (compare these with the boundary conditions that the function u satisfies). Clearly, there is only one such linear function ℓ, namely, $\ell(x) = \frac{5}{\pi}x$. Derive the BVP satisfied by the function $v(x,t)$ – you will obtain a BVP with homogeneous (i.e., zero) Dirichlet BCs. Be careful – the PDE for v may be different than the PDE for u, and the IC for v will certainly be different from the one for u.

(b) Solve the BVP for v derived in part (a). You again may use the expressions for the solutions of BVPs for the heat equation given in Problem 1 (without deriving them). Also, you may use the fact that the sine Fourier series of the function $f(x) = x$ for $x \in [0,L]$ is

$$\frac{2L}{\pi}\left(\sin\frac{\pi x}{L} - \frac{1}{2}\sin\frac{2\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} - \frac{1}{4}\sin\frac{4\pi x}{L} + \cdots\right) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\frac{n\pi x}{L}$$

(this expression is derived in Example 1 on pages 600–601 of the book).

(c) Having solved part (b), write down the solution $u(x,t)$ of the original BVP.

Problem 3. In this problem you will make some predictions about the asymptotic behavior (i.e., when $t \to \infty$) of the solution $u(x,t)$ of the boundary value problem

$$\begin{aligned} &u_t = \alpha^2\, u_{xx} + \psi(x) \ , \qquad x \in [0,L] \ , \quad t \in [0,\infty) \\ &u(0,t) = 0 \ , \qquad u(L,t) = 0 \qquad \text{for } t \in [0,\infty) \\ &u(x,0) = f(x) \qquad \text{for } x \in [0,L] \ . \end{aligned}$$

Physically, this problem describes the temperature distribution in a rod of length L with insulated side walls and ends at $x = 0$ and $x = L$ kept at zero temperature. The initial temperature in the rod is given by the function $f(x)$ and, more interestingly, there are sources of heat in the rod whose power is given by the function $\psi(x)$ in the PDE.

One can solve this problem completely (which you will do in Problem 4 below), but before doing this, try to obtain some information about the behavior of the solution $u(x,t)$ at large times. Since the temperatures at the ends of the rod do not depend on time, and the intensity of the sources of heat is time-independent as well, it is clear that after some initial period of more or less rapid changes, the solution $u(x,t)$ will tend to some time-independent function. Let us call this function $u_\infty(x)$:

$$u_\infty(x) := \lim_{t\to\infty} u(x,t) \ .$$

Since this function does not depend on t, it will be a solution of some *ordinary* differential equation!

(a) From the PDE given in this problem, obtain an ODE for the function $u_\infty(x)$.

(b) From the BCs for $u(x,t)$, obtain BCs for $u_\infty(x)$. Note that the initial condition for $u(x,t)$ will not matter in the limit $t \to \infty$.

(c) Solve the boundary value problem for the asymptotic temperature distribution $u_\infty(t)$ in the case $\alpha = 1$, $L = \pi$, $\psi(x) = 2\sin 5x$, $f(x) = \sin 3x$.

(d) Sketch the function $u_\infty(x)$. Find the highest and the lowest temperatures in the rod after very long time.

Problem 4. Now you will find the solution of the boundary value problem

$$\begin{aligned}
&u_t = \alpha^2\, u_{xx} + \psi(x) \ , \qquad x \in [0,L] \ , \quad t \in [0,\infty) \\
&u(0,t) = 0 \ , \qquad u(L,t) = 0 \qquad \text{for } t \in [0,\infty) \\
&u(x,0) = f(x) \qquad \text{for } x \in [0,L] \ .
\end{aligned}$$

This is the same as in Problem 3, but there you only found the asymptotic behavior of $u(x,t)$ as $t \to \infty$, while here you will solve the problem completely.

(a) Because of the boundary conditions, look for a solution of the problem of the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\frac{n\pi x}{L} \ .$$

Assume that the function $\psi(x)$ in the right-hand side of the PDE can be expanded in a sine Fourier series as

$$\psi(x) = \sum_{n=1}^{\infty} \psi_n \sin\frac{n\pi x}{L} \ ,$$

where the coefficients ψ_n are given by the standard formula, $\psi_n = \dfrac{2}{L}\displaystyle\int_0^L \psi(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x$.

Plug these expansions in the partial differential equation to show that the functions $T_n(t)$ satisfy the non-homogeneous ODEs

$$T_n'(t) + \left(\frac{\alpha n\pi}{L}\right)^2 T_n(t) = \psi_n \ .$$

(b) Assume that the sine Fourier series of the the initial condition $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\frac{n\pi x}{L} \ .$$

Plug the expansion of $u(x,t)$ into the initial condition to show that the initial conditions for the functions $T_n(t)$ are $T_n(0) = f_n$.

(c) Solve the initial value problems for the functions $T_n(t)$ derived in parts (a) and (b).

(d) Using your results from parts (a) and (c), write down the solution $u(x,t)$ of the original boundary value problem.

(e) Write down the solution $u(x,t)$ of the original boundary value problem in the case $\alpha = 1$, $L = \pi$, $\psi(x) = 2\sin 5x$, $f(x) = \sin 3x$ (the same as in Problem 4(c) above).

(f) Check if the asymptotic (i.e., as $t \to \infty$) behavior of the solution $u(x,t)$ obtained in part (e) is the same as the function $u_\infty(x)$ obtained in Problem 4(d).