MATH 3413 Problems assigned on 4/1/14

Sec. 9.1: problems 2, 9, 10, 11, 13.

Sec. 9.2: problems 17, 24(b).

Hint for Problem 9.2/17: You may use the integrals

$$\int t \sin at \, \mathrm{d}t = \frac{1}{a^2}\left(\sin at - at\cos at\right) \ , \qquad \int t \cos at \, \mathrm{d}t = \frac{1}{a^2}\left(\cos at + at\sin at\right) \ .$$

Sec. 9.3: problems 2, 17.

Hint for Problem 9.3/2: The hint for Problem 9.2/17 will be useful.

Additional problem 1. Let $\mathbb{R}^2$ stand for the vector space of all two-dimensional vectors, $\mathbf{u} = \binom{u_1}{u_2}$. Let the inner product in $\mathbb{R}^2$ be given by

$$\langle \mathbf{u}, \mathbf{v} \rangle := \sum_{i=1}^{2} \sum_{j=1}^{2} u_i \, a_{ij} \, v_j \ ,$$

where $a_{11} = 2$, $a_{12} = a_{21} = 1$, $a_{22} = 4$.

Let $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ be a basis in $\mathbb{R}^2$, where $\mathbf{v}^{(1)} = \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and $\mathbf{v}^{(2)} = \begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

(a) Check that $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ is an orthogonal basis, i.e., that the inner product of $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ is zero: $\langle \mathbf{v}^{(1)}, \mathbf{v}^{(2)} \rangle = 0$.

(b) Find $\langle \mathbf{v}^{(1)}, \mathbf{v}^{(1)} \rangle$ and $\langle \mathbf{v}^{(2)}, \mathbf{v}^{(2)} \rangle$.

(c) If $\mathbf{u} = \begin{pmatrix} 9 \\ -1 \end{pmatrix} = \alpha_1 \mathbf{v}^{(1)} + \alpha_2 \mathbf{v}^{(2)}$, then find the components α_1 and α_2 of $\mathbf{u}$ in the basis $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ by solving the system of linear equations for them coming from

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \alpha_1 \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} + \alpha_2 \begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \end{pmatrix} \ .$$

(d) Independently of part (c), if $\mathbf{u} = \begin{pmatrix} 9 \\ -1 \end{pmatrix} = \alpha_1 \mathbf{v}^{(1)} + \alpha_2 \mathbf{v}^{(2)}$, find the components α_1 and α_2 by using the formula (which relies on the <u>orthogonality</u> of the basis $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$)

$$\alpha_j = \frac{\langle \mathbf{u}, \mathbf{v}^{(j)} \rangle}{\langle \mathbf{v}^{(j)}, \mathbf{v}^{(j)} \rangle} \ .$$

(e) The basis $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ is orthogonal, but not orthonormal. From the basis $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ construct an orthonormal basis $\{\widetilde{\mathbf{v}}^{(1)}, \widetilde{\mathbf{v}}^{(2)}\}$ of the form $\widetilde{\mathbf{v}}^{(i)} = \beta_i \mathbf{v}^{(i)}$, where β_i are appropriately chosen constants.

Additional problem 2. Let f be a periodic function of period 2π which for t between $-\pi$ and π is defined as

$$f(t) = \begin{cases} 0\,, & -\pi < t \le 0 \\ t\,, & 0 < t \le \pi\,; \end{cases}$$

the graph of f is sketched in Figure 1.

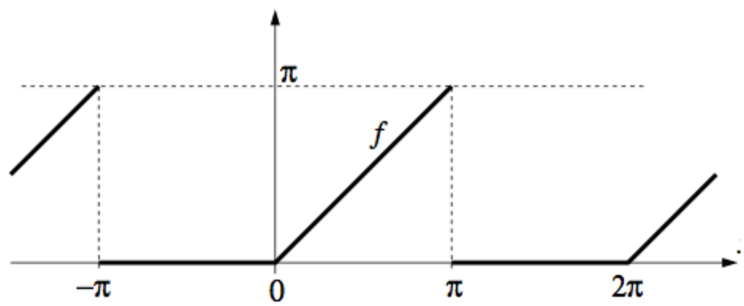

Figure 1: The graph of the function f.

The Fourier series of f is the following (you do <u>not</u> have to prove this!):

$$\begin{aligned} f(t) &= \frac{\pi}{4} - \frac{2}{\pi}\left(\cos t + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2} + \frac{\cos 7t}{7^2} + \cdots\right) \\ &\quad + \sin t - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \frac{\sin 4t}{4} + \cdots\,. \end{aligned}$$

Let the function g be a periodic function of period 2π sketched in Figure 2. Write $g(t)$ in terms of $f(t)$. Using the Fourier series of f, find the Fourier series of g.

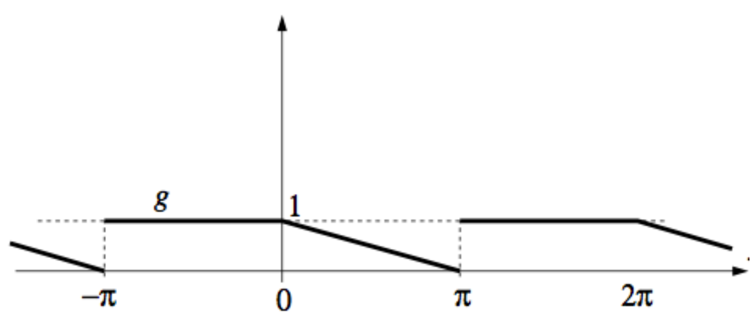

Figure 2: The graph of the function g.