Homogeneous linear ODE of order n with constant coefficients

We consider the equation

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0, \qquad (H)$$

where $a_0, \ldots, a_n$ are real constants, and $a_n \neq 0$.

Let (C) stand for the characteristic equ:

$$a_n r^n + \cdots + a_1 r + a_0 = 0. \qquad (C)$$

The characteristic equation has exactly n roots (in general complex), if we count each root with its multiplicity. If all coefficients $a_0, \ldots, a_n$ are real, the complex roots come in conjugate pairs, i.e., if $\alpha + i\beta$ is a root of (C) of multiplicity p, then $\alpha - i\beta$ is also a root of (C) of multiplicity p.

Example: Consider the 13^{th} degree eqn

$$(r-3)^4 (r+1)^5 (r^2+4)^2 = 0:$$

- 3 is a root of multiplicity 4
- -1 is a root of multiplicity 5
- $\pm 2i$ are roots, each of mult. 2

Each root of the characteristic equation (C) contributes a term to the general solution of the homogeneous equation (H). Namely:

- Each real root $r_1 \in \mathbb{R}$ of (C) of multiplicity p contributes a term
$$Q_{p-1}(x)\, e^{r_1 x}$$
to the general solution of (H); here $Q_{p-1}(x)$ is a polynomial of degree $p-1$ (having exactly p coefficients:
$$Q_p(x) = C_1 + C_2 x + C_3 x^2 + \dots + C_p x^{p-1}.$$

- Each conjugate pair of complex roots $\alpha \pm i\beta$ of (C), each of the two roots with multiplicity p, contributes a term
$$e^{\alpha x}\left[R_{p-1}(x) \cos \beta x + S_{p-1}(x) \sin \beta x\right],$$
where $R_{p-1}(x)$ and $S_{p-1}(x)$ are arbitrary polynomials of degree $p-1$:
$$R_{p-1}(x) = A_1 + A_2 x + \dots + A_p x^{p-1},$$
$$S_{p-1}(x) = B_1 + B_2 x + \dots + B_p x^{p-1}.$$

Example:

$$9y^{(5)} - 6y^{(4)} + y^{(3)} = 0$$

$$9r^5 - 6r^4 + r^3 = 0$$

$$r^3(9r^2 - 6r + 1) = 0$$

$$9r^3\left(r - \tfrac{1}{3}\right)^2 = 0$$

$$\Rightarrow \begin{cases} \bullet\ 0 \text{ is a root of mult. } 3 \\ \bullet\ \tfrac{1}{3} \text{ is a root of mult. } 2 \end{cases}$$

$\Rightarrow$ the general solution of the ODE is

$$y(x) = \underbrace{C_1 + C_2 x + C_3 x^2}_{\text{coming from the root } 0} + \underbrace{(C_4 + C_5 x)\, e^{\frac{1}{3}x}}_{\text{coming from the root } \frac{1}{3}}$$

Example:

$$(D-2)^4(D^2 - 6D + 25)^3 y = 0$$

$$(r-2)^4(r^2 - 6r + 25)^3 = 0;$$

the roots of $r^2 - 6r + 25 = 0$ are $3 \pm 4i$ (each with multiplicity 1), so the charact. eqn can be written as

$$(r-2)^4\,[r-(3+4i)]^3\,[r-(3-4i)]^3=0,$$

and the general solution of the ODE is

$$y(x)=\underbrace{\left(C_1+C_2x+C_3x^2+C_4x^3\right)e^{2x}}_{\text{coming from the root 2}}$$

$$\underbrace{+\,e^{3x}\Big[\left(C_5+C_6x+C_7x^2\right)\cos 4x+\left(C_8+C_9x+C_{10}x^2\right)\sin 4x\Big]}_{\text{coming from the conjugate pair } 3\pm 4i}$$

<u>Example:</u>

$$y^{(11)}+12y^{(9)}+48y^{(7)}+64y^{(5)}=0,$$

which can be written as

$$D^5\left(D^2+4\right)^3y=0,$$

so the charact. equ. is

$$r^5\left(r^2+4\right)^3=0,$$

and the general solution is

$$y(x)=C_1+C_2x+C_3x^2+C_4x^3+C_5x^4$$
$$+\left(C_6+C_7x+C_8x^2\right)\cos 2x$$
$$+\left(C_9+C_{10}x+C_{11}x^2\right)\sin 2x$$

(<u>why?</u>).