General solution of a linear homogeneous ODE with constant coefficients:

To find the general solution of the linear homogeneous ODE with constant coefficients $Ly = 0$, where

$$Ly := a_n\, y^{(n)} + a_{n-1}\, y^{(n-1)} + \cdots + a_1\, y' + a_0\, y \ , \tag{1}$$

first solve the characteristic equation of this ODE. Each root of the characteristic equation contributes a term to the general solution of the ODE:

- each real root r of the characteristic equation of multiplicity p contributes to the general solution a term
$$P_{p-1}(x)\, e^{rx} \ ,$$
where $P_{p-1}(x)$ is a polynomial of degree $(p-1)$;
- each conjugate pair of complex roots $\alpha \pm i\beta$ of the characteristic equation, where each of the the two roots has multiplicity p, contributes to the general solution a term
$$e^{\alpha x}\left[Q_{p-1}(x)\cos\beta x + R_{p-1}(x)\sin\beta x\right] \ ,$$
where $Q_{p-1}(x)$ and $R_{p-1}(x)$ are polynomials of degree $(p-1)$.

Particular solutions of a linear non-homogeneous ODE with constant coefficients

The general solution of the linear non-homogeneous ODE with constant coefficients $Ly = f(x)$ (where Ly is given by the expression (1) above) is equal to the sum of the general solution of the associated homogeneous equation $Ly = 0$ and a particular solution of $Ly = f(x)$. First solve the characteristic equation of $Ly = 0$ and find the general solution of $Ly = 0$, and then find a particular solution of $Ly = f(x)$ by doing the following:

- in the case $f(x) = e^{cx}\, P_m(x)$, if c is a root of the characteristic equation of $Ly = 0$ with multiplicity s, then look for a particular solution $y_p(x)$ of $Ly = f(x)$ of the form
$$y_p(x) = x^s\, e^{cx}\, Q_m(x) \ ,$$
where $Q_m(x)$ is a polynomial of degree m;
- in the case $f(x) = e^{cx}\left[P_{m_1}(x)\cos dx + \widetilde{P}_{m_2}(x)\sin dx\right]$, if $c + id$ is a root of the characteristic equation of $Ly = 0$ with multiplicity s, then define $m := \max(m_1, m_2)$, and look for a particular solution $y_p(x)$ of $Ly = f(x)$ of the form
$$y_p(x) = x^s\, e^{cx}\left[Q_m(x)\cos dx + \widetilde{Q}_m(x)\sin dx\right] \ ,$$
where $Q_m(x)$ and $\widetilde{Q}_m(x)$ are polynomials of degree m.

If the equation has the form $Ly = f_1(x) + f_2(x)$, its general solution is a sum of the general solution of $Ly = 0$ and the particular solutions of the equations $Ly = f_1(x)$ and $Ly = f_2(x)$.