Example - flow of an ODE

Consider the IVP

$$\begin{cases} \dot{x} = -x \\ \dot{y} = x^2 + y \end{cases} \qquad \begin{cases} x(0) = x^{(0)} \\ y(0) = y^{(0)}. \end{cases}$$

Solve the first ODE first:

$$\int \frac{dx}{x} = -\int dt \Rightarrow x(t) = C_1 e^{-t}.$$

Plug $x(t)$ in the 2nd eqn: $\dot{y} - y = x^2 = C_1^2 e^{-2t}$.

This is a linear 1^{st} order ODE of the form

$$\dot{y} + P(t)\, y = Q(x).$$

To solve such an equation, first find the integrating factor

$$\mu(t) := e^{\int P(t)\,dt} = e^{\int (-1)\,dt} = e^{-t}.$$

Then multiply the ODE by $\mu(t)$, and notice that

$$\frac{d}{dt}\left[\mu(t) y(t)\right] = \mu(t)\dot{y}(t) + P(t)\mu(t) y(t)$$

$$= \mu(t)\left[\dot{y}(t) + P(t)\, y(t)\right]$$

because, by the chain rule,

$$\frac{d}{dt} e^{\int P(t)\,dt} = e^{\int P(t)\,dt} \frac{d}{dt}\int P(t)\,dt = \mu(t) P(t).$$

Therefore, the ODE multiplied by $\mu(t)$ becomes

$$\frac{d}{dt}\left[\mu(t) y(t)\right] = Q(t)\,\mu(t),$$

which in our case becomes

$$\frac{d}{dt}\left[e^{-t}y(t)\right] = C_1^2 e^{-2t}\cdot e^{-t} = C_1 e^{-3t}.$$

Integrate both sides w.r.t. t:

$$e^{-t}y(t) = -\frac{1}{3}C_1^2 e^{-3t} + C_2$$

$$\Rightarrow y(t) = -\frac{1}{3}C_1^2 e^{-2t} + C_2 e^{t}.$$

Imposing the ICs:

$$\begin{cases} x^{(0)} = x(0) = C_1 \\ y^{(0)} = y(0) = -\frac{1}{3}C_1^2 + C_2 \end{cases} \Longrightarrow \begin{cases} C_1 = x^{(0)} \\ C_2 = y^{(0)} + \frac{1}{3}x^{(0)2} \end{cases}$$

Therefore the solution of the IVP is

$$\begin{cases} x(t) = x^{(0)} e^{-t} \\ y(t) = -\frac{1}{3}x^{(0)2} e^{-2t} + \left(y^{(0)} + \frac{1}{3}x^{(0)2}\right)e^{t}. \end{cases}$$

We can write the flow as

$$\underline{\varphi}_t(\underline{x}^{(0)}) = \begin{pmatrix} x_1^{(0)} e^{-t} \\ -\frac{1}{3}x_1^{(0)2} e^{-2t} + \left(x_2^{(0)} + \frac{1}{3}x_1^{(0)2}\right)e^{t} \end{pmatrix},$$

where we changed the notations from (x,y) to $(x_1, x_2) = \underline{x}$.

Exercise: Check the semigroup property, $\underline{\varphi}_t(\underline{\varphi}_s(\underline{x}^{(0)})) = \underline{\varphi}_{t+s}(\underline{x}^{(0)})$ for this flow.