Conditions for $f(z) = u(x,y) + i\,v(x,y)$ to be differentiable

The function f is differentiable at $z_0 \in \mathbb{C}$ if the limit

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists and has the same value no matter how Δz tends to 0 (in the complex plane Δz can go to 0 "horizontally", "vertically", or in any other way).

1) Let $\Delta z \to 0$ "horizontally", i.e.,

$$\Delta z = \Delta x \to 0, \qquad \Delta x \in \mathbb{R}.$$

Then

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} =$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \Big[\big(u(x_0+\Delta x, y_0) + i\,v(x_0+\Delta x, y_0) \big) - \big(u(x_0,y_0) + i\,v(x_0,y_0) \big) \Big]$$

$$= \lim_{\Delta x \to 0} \left[\frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} + i\, \frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x} \right]$$

$$= u_x(x_0,y_0) + i\, v_x(x_0,y_0) \qquad (*)$$

2) let $\Delta z \to 0$ "vertically", i.e.,

$\Delta z = i \Delta y$, $\Delta y \in \mathbb{R}$, $\Delta y \to 0$.

Then

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta y \to 0} \frac{1}{i\Delta y} \Big[\big(u(x_0, y_0+\Delta y) + i\, v(x_0, y_0+\Delta y) \big) - \big(u(x_0,y_0) + i\,v(x_0,y_0) \big) \Big]$$

$$= \lim_{\Delta y \to 0} \left[\frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{i \Delta y} + \cancel{i} \, \frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{\cancel{i} \Delta y} \right]$$

$$= \frac{1}{i} u_y(x_0, y_0) + v_y(x_0, y_0)$$

$$= v_y(x_0, y_0) - i u_y(x_0, v_0) \qquad (**)$$

Comparing $(*)$ and $(**)$, we see that

$$\underline{u_x} + i v_x = \underline{v_y} - i u_y$$

$$\Longrightarrow \quad u_x = v_y, \quad v_x = -u_y.$$

<u>Conclusion:</u> If $f = u + iv$ is differentiable at $z_0 = x_0 + i y_0$, then

$$\left| \begin{array}{l} u_x(x_0, y_0) = v_y(x_0, y_0) \\ u_y(x_0, y_0) = -v_x(x_0, y_0) \end{array} \right.$$

— <u>Cauchy-Riemann equations.</u>