Elements of Counting

The Basic Principle of Counting. Suppose that two experiments are to be performed. If experiment 1 can result in any one of m possible outcomes, and if for each outcome of experiment 1 there are n possible outcomes of experiment 2, then there are mn possible outcomes of the two experiments.
Proof: Simply enumerate all possible outcomes:

Exp 2 $\rightarrow$ Exp 1 $\downarrow$	1	2	$\ldots$	n
1	$(1,1)$	$(1,2)$	$\ldots$	$(1,n)$
2	$(2,1)$	$(2,2)$	$\ldots$	$(2,n)$
$\vdots$				
m	$(m,1)$	$(m,2)$	$\ldots$	(m,n)

Example: Drawing one card as a sequence of two experiments:

Exp 2 $\rightarrow$ Exp 1 $\downarrow$	$\spadesuit$	$\clubsuit$	$\diamondsuit$	$\heartsuit$
2	2$\spadesuit$	2$\clubsuit$	2$\diamondsuit$	2$\heartsuit$
3	3$\spadesuit$	3$\clubsuit$	3$\diamondsuit$	3$\heartsuit$
4	4$\spadesuit$	4$\clubsuit$	4$\diamondsuit$	4$\heartsuit$
5	5$\spadesuit$	5$\clubsuit$	5$\diamondsuit$	5$\heartsuit$
6	6$\spadesuit$	6$\clubsuit$	6$\diamondsuit$	6$\heartsuit$
7	7$\spadesuit$	7$\clubsuit$	7$\diamondsuit$	7$\heartsuit$
8	8$\spadesuit$	8$\clubsuit$	8$\diamondsuit$	8$\heartsuit$
9	9$\spadesuit$	9$\clubsuit$	9$\diamondsuit$	9$\heartsuit$
10	10$\spadesuit$	10$\clubsuit$	10$\diamondsuit$	10$\heartsuit$
J	J$\spadesuit$	J$\clubsuit$	J$\diamondsuit$	J$\heartsuit$
Q	Q$\spadesuit$	Q$\clubsuit$	Q$\diamondsuit$	Q$\heartsuit$
K	K$\spadesuit$	K$\clubsuit$	K$\diamondsuit$	K$\heartsuit$
A	A$\spadesuit$	A$\clubsuit$	A$\diamondsuit$	A$\heartsuit$

Generalization: We are to perform r experiments with: n_1 possible outcomes of Exp 1; n_2 possible outcomes of Exp 2 $\forall$ outcome of Exp 1; n_3 possible outcomes of Exp 3 $\forall$ outcome of Exp 1 and Exp 2; $\ldots$; n_r possible outcomes of Exp r $\forall$ outcome of Exp 1, Exp 2, $\ldots$, Exp $(r-1)$. Then there is a total of $n_1\, n_2\, n_3 \cdots n_r$ possible outcomes of the r experiments.

Example: A committee consists of 3 freshmen, 4 sophomores, 5 juniors, and 2 seniors. A subcommittee of four people, one person from each class, is to be chosen. How many different subcommittees are possible? *Answer:* $3 \cdot 4 \cdot 5 \cdot 2 = 120$.

Example: How many non-negative numbers can be represented in the binary system by 1 byte, i.e., an 8-tuple of 0's and 1's? $\boxed{0/1}\boxed{0/1}\boxed{0/1}\boxed{0/1}\boxed{0/1}\boxed{0/1}\boxed{0/1}\boxed{0/1}$ *Answer:* $2^8 = 256$.

Example: How many different 7-place license plates are possible if the 3 places are to be occupied by letters, and the next 4 places by numbers. *Answer:* $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 175,760,000$.

Example: How many different 7-place license plates are possible if the 3 places are to be occupied by letters, and the next 4 places by numbers, if repetition among letters or numbers were prohibited? *Answer:* $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 78,624,000$.

Permutations – *ordered* arrangements of *distinct* objects.

Theorem: The number of permutations of n distinct objects is $n! := n \cdot (n-1) \cdots 2 = \cdot 1$.

Terminology:

- distinct = different = distinguishable
- *n-tuple* = an *ordered* set of n objects.
- *Group* = an *unordered* set of n objects.

Example: The number of different ordered arrangements of the numbers 1, 2, 3 is $3! = 6$, namely, (123), (132), (213), (231), (312), (321).

Example: Number of distinct rankings in a class of 6 men and 4 women. (Assuming that all grades are different.)

- # of different rankings $= 10! = 3,628,800$.
- If the men are ranked among themselves, and the women are ranked among themselves:

$$\begin{aligned} 6! &= 720 \text{ different rankings of the men,} \\ 4! &= 24 \text{ different rankings of the women,} \\ 6!\,4! &= 17,280 \text{ different rankings.} \end{aligned}$$

Permutations of possibly indistinguishable objects

Example: How many different letter arrangements can be formed by using the letters
P E P P E R ?

Solution: If all the letters were distinguishable, $P_1\ E_1\ P_2\ P_3\ E_2\ R$, there would be $6! = 720$ arrangements.
However, not all of these arrangements are different if the P's and the E's don't have subscripts! How many are the *different* arrangements?
For each configuration of letters (without subscripts), say P R E P P E, there are 3! possible permutations of the letters P among themselves, and 2! permutations of the letters E among themselves,

$$\begin{array}{cccccc} P_1 & R & E_1 & P_2 & P_3 & E_2 \\ P_1 & R & E_1 & P_3 & P_2 & E_2 \\ P_2 & R & E_1 & P_1 & P_3 & E_2 \\ P_2 & R & E_1 & P_3 & P_1 & E_2 \\ P_3 & R & E_1 & P_1 & P_2 & E_2 \\ P_3 & R & E_1 & P_2 & P_1 & E_2 \end{array} \qquad \begin{array}{cccccc} P_1 & R & E_2 & P_2 & P_3 & E_1 \\ P_1 & R & E_2 & P_3 & P_2 & E_1 \\ P_2 & R & E_2 & P_1 & P_3 & E_1 \\ P_2 & R & E_2 & P_3 & P_1 & E_1 \\ P_3 & R & E_2 & P_1 & P_2 & E_1 \\ P_3 & R & E_2 & P_2 & P_1 & E_1 \end{array}$$

thus, the total number of different letter arrangements is $\dfrac{6!}{3!\,2!} = 60$.

Generalization: The number of permutations of n objects, of which n_1 are alike, n_2 are alike, $\ldots$, n_r are alike is

$$\frac{n!}{n_1!\,n_2!\,\cdots\,n_r!} =: \binom{n}{n_1,\, n_2,\, n_3,\, \cdots\, ,\, n_r}\,.$$

Note that $n_1 + n_2 + \cdots + n_r = n$.

Example: How many different linear arrangements are there of the letters A, B, C, D, E, F, for which:

- there are no restrictions?
- A and B are next to each other?
- A is before B? (Not necessarily directly before B.)
- E is not last in line?

Combinations = different groups of r objects that can be formed from a total of n distinguishable objects?

A reminder: A "group" means an *unordered collection* of objects.

Example: How many different groups of 3 can be selected from the 5 items A, B, C, D, and E?

Solution: 5 ways the select the initial item, 4 ways to select the next item, and 3 ways to select the last item. But, since the order does not matter, the 3! selections ABC, ACB, BAC, BCA, CAB, and CBA, correspond to the same *group* of objects chosen. That is, we have *overcounted* by a factor of $3! = 6$. Hence, the number of *groups* of 3 objects drawn from 5 distinguishable objects is $\frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3} = 10$.

Generalization: Counting the number of *combinations*, i.e., different groups of r objects that can be formed from a total of n distinguishable objects:

- the number of different *ordered* selections is $\underbrace{n\,(n-1)\,(n-2)\,\cdots\,(n-r+1)}_{r \text{ factors}}$;
- since $r!$ ordered selections correspond to *one group*, the number of groups is $\frac{n\,(n-1)\,(n-2)\,\cdots\,(n-r+1)}{r!}$, which is equal to

$$\frac{n\,(n-1)\,\cdots\,(n-r+1)}{r!}\cdot\frac{(n-r)\,(n-r-1)\,\cdots\,2\cdot 1}{(n-r)\,(n-r-1)\,\cdots\,2\cdot 1} = \frac{n!}{(n-r)!\,r!}\ .$$

Binomial coefficients: For $0 \le r \le n$, define the *binomial coefficient* $\binom{n}{r}$ (read "n choose r") by

$$\binom{n}{r} := \frac{n!}{(n-r)!\;r!}\ ,$$

where, by definition, $0! = 1$, so that $\binom{n}{0} = \binom{n}{n} = 1$.

Example: Number of groups of 4 cards drawn from a deck of 52 cards: $\binom{52}{4} = \frac{52\cdot 51\cdot 50\cdot 49}{4!} = 270,725$.

Example: From a group of 8 women and 6 men, a committee consisting of 4 women and 3 men is to be formed.

- How many different committees are possible?
- What if one man and one woman refuse to serve together?

Example: A student is to answer 7 out of 10 questions in an examination.

- How many choices does she have? <u>Answer</u>: $\binom{10}{7}$.
- How many if she must answer at least 3 of the first 5 questions?

The binomial theorem:

$$(x+y)^n = \sum_{r=0}^{n}\binom{n}{r}\,x^r\,y^{n-r}\ .$$

An elementary identity: $\displaystyle\binom{n}{r} = \binom{n}{n-r}\,, \qquad 1 \le r \le n.$

Direct proof: $\displaystyle\binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \binom{n}{n-r}.$

Combinatorial proof: The number $\displaystyle\binom{n}{r}$ is equal to the number of different way of choosing a group of r objects out of n different objects. But choosing the r objects in the group is equivalent to choosing the $n-r$ objects that do *not* belong to the group, which can be done in $\displaystyle\binom{n}{n-r}$ different ways.

Another identity: $\displaystyle\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ for $1 \le r \le n$.

Multinomial coefficients: If $n_1 + n_2 + \cdots + n_r = n$, then

$$\binom{n}{n_1, n_2, \cdots, n_r} = \frac{n!}{n_!!\,n_2!\,\cdots n_r!}\ .$$

Theorem: The number of ways to divide a set of n distinct objects into r distinct groups of respective sizes n_1, n_2, ..., n_r (like in the figure below) is $\displaystyle\binom{n}{n_1, n_2, \cdots, n_r}$.

$$\underbrace{\underbrace{\bullet\bullet\cdots\bullet}_{n_1 \text{ objects}}\ \underbrace{\bullet\bullet\cdots\bullet}_{n_2 \text{ objects}}\ \quad\cdots\quad\ \underbrace{\bullet\bullet\cdots\bullet}_{n_r \text{ objects}}}_{n_1+n_2+\cdots+n_r=n \text{ objects}}$$

Proof: There are: $\binom{n}{n_1}$ distinct ways to choose for the 1st group; having chosen the 1st group of n_1 elements, there are $\binom{n-n_1}{n_2}$ distinct choices for the 2nd group; having chosen the 1st and the 2nd groups of $n_1 + n_2$ elements total, there are $\binom{n-n_1-n_2}{n_3}$ distinct choices for the 3rd group, ..., finally, there are $\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r}$ choices for the rth group. Apply the generalized principle of counting and do the necessary cancellations.

The multinomial theorem:

$$(x_1 + x_2 + \cdots + x_r)^n = \sum \binom{n}{n_1, n_2, \cdots, n_r} x_1^{n_1}\, x_2^{n_2}\, \cdots\, x_r^{n_r}\ ,$$

where the summation is over all ordered sets $(n_1, n_2, \ldots, n_r)$ such that $0 \le n_1 \le n$, $0 \le n_2 \le n$, ..., $0 \le n_r \le n$, and $n_1 + n_2 + \cdots + n_r = n$.

Example: Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes take place?