Analysis of Population Modeling

Armand Ghosh
Adviser: Nikola Petrov

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1 Introduction

This article is a summary of my studies, with Professor Nikola Petrov, analyzing singularities, folds, bifurcations, cusps, and corners of graphs reflecting a typical population model over time. In our studies, we utilize ordinary and partial differential equations and multivariable calculus to examine how the size of a population behaves with respect to reproduction and external stimuli.

We will first look at how a population grows with respect to carrying capacity of a habitat (e.g. land, water, resources). Later, we will take into consideration the predator/prey impact of the population (e.g. a species of fish being consumed by bears), assuming that the population we are examining has a predatory species. We defined our population rate by a differential equation with respect to time t, taking into consideration of the first two of three impacts a population might have.

2 Function Modeling using Differential Equations

2.1 Population Model Differential Equation Solving

$$\frac{dx}{dt} = \alpha x \left(1 - \frac{x}{\kappa}\right) \tag{1}$$

The equation we formulated contains the time variable t, the population variable x, and two adjustable constants α and κ (known by their Greek letters alpha and kappa), which are different per population. α reflects the size of the population growth, while $\frac{\alpha}{\kappa}$ reflects the population's decline due to the effect of its carrying capacity. Solving for the population variable (which requires integration), we obtain a family of logistic functions, denoted as $x(t)$.

$$x(t) = \frac{-\kappa C e^{\alpha t}}{1 - Ce^{\alpha t}} \tag{2}$$

The only thing that is preventing our population model from being a single function is the constant C that is formed from the process of integration. To fix this problem, we need a starting point. We need a starting time t_0 and our population at that time: $x_0 = x(t_0)$. In general, our starting time t_0 would equal 0, or the current time, and the remaining values of t correspond to the number of time units (e.g. years, months, hours, etc.) after or before the current time. Substituting $t_0 = 0$ for t and $x(0) = x_0$ for $x(t)$ and solving for C in *Equation 2*, we get

$$C = \frac{x_0}{x_0 - \kappa}$$

$$x(t) = \frac{-\kappa \frac{x_0}{x_0 - \kappa} e^{\alpha t}}{1 - \frac{x_0}{x_0 - \kappa} e^{\alpha t}} = \frac{\kappa x_0 e^{\alpha t}}{x_0(e^{\alpha t} - 1) + \kappa}$$

2.2 Phase Portraits

In a logistical equation, the values of x_0 determine the shape of the graph and its behavior as time increases. For $0 < x_0 < \kappa$, the graph behaves similarly to one of an arctangent function, as shown in *Figure 1*. Here, $\lim_{t \to -\infty} x(t) = 0$ and $\lim_{t \to \infty} x(t) = \kappa$. In other words, $x(t)$ converges both ways.

There are two conditions that exhibit divergence in one of the directions: negative or positive. If $x_0 > \kappa$, then the graph converges to κ as t approaches $+\infty$, but the graph diverges when t approaches $-\infty$. At the same time, $x(t)$ becomes infinitely large. If $x_0 < 0$, then the opposite is true, with $x(t)$ converging to 0 as t approaches $-\infty$ and $x(t)$ negatively diverging as t approaches ∞.

The remaining two conditions, $x_0 = 0$ and $x_0 = \kappa$, causes the graph of $x(t)$ to be horizontal in both cases at 0 and κ respectively. The only time when the graph does not exist is when the denominator of $x(t)$ is equal to 0. The only case here is when x_0 and κ are both equal to 0.

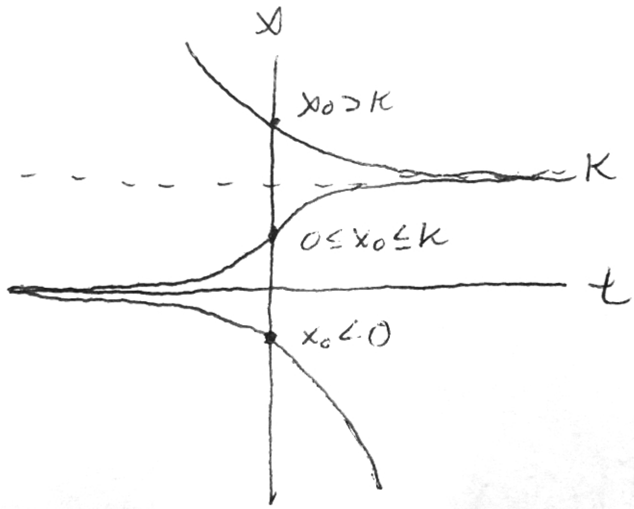

Figure 1: The Logistic Equation $x(t)$, which shows the varying behaviors of x when changing the value of x_0.

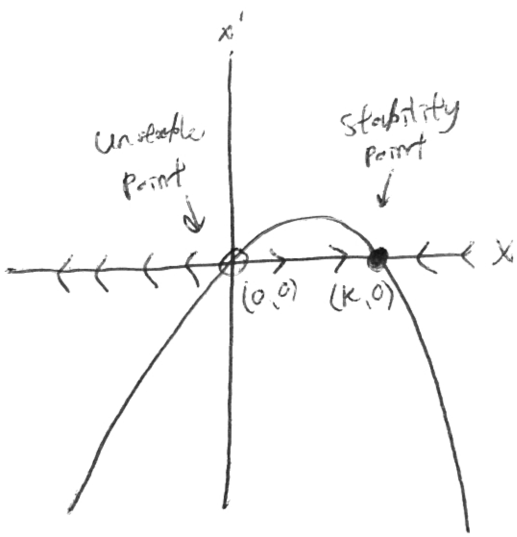

Figure 2: The phase portrait of *Equation 1*

The affect that x_0 has on the tendency of the graph $x(t)$ is best illustrated as a phase portrait, shown in *Figure 2*.

Steven H. Strogatz's *Nonlinear Dynamics and Chaos* [1] illustrates that, given a differential equation $\frac{dx}{dt}$, we can predict the behavior of each graph within a collection of graphs. A phase portrait displays, given a starting point x at $t = 0$, the direction $x(t)$ heads when t increases, without having to solve for $x(t)$. This method can forecast the manner of how a graph converges to a value or diverges, by simply graphing the differential equation $\frac{dx}{dt}$ with respect to x. $\frac{dx}{dt}$ is known as the trajectory of the phase portrait. The critical points (where the direction of the phase portrait changes) are located whenever $\frac{dx}{dt} = 0$. Whenever $\frac{dx}{dt} > 0$, then the particle on the trajectory moves to the right until the particle approaches near a critical point. Likewise, whenever $\frac{dx}{dt} < 0$, the particle of the trajectory moves leftward, until the particle approaches a critical point.

The critical points of the function characterize how a particle behaves if it is near the point. There are four types of critical points:

(1) Stable: Attracts particle. $\frac{dx}{dt} > 0$ to the left of the particle, but $\frac{dx}{dt} < 0$ to the right of the particle.

(2) Unstable: Repels particle. Opposite conditions of stable particles.

(3) Half-stable (left to right): Attracts the particle from the left and repels the particle from the right. Mathematically, $\frac{dx}{dt} > 0$ both to the left and right of the critical point.

(4) Half-stable (right to left): Repels the particle from the right and attracts the particle from the left. Mathematically, $\frac{dx}{dt} < 0$ both to the left and right of the critical point.

The stability is illustrated in *Figures 2 and 3*. *Figure 2* illustrates conditions (1) and (2) of the critical points, whereas *Figure 3* illustrates conditions (1), (3), and (4) of the critical points.

2.3 Adding the Predation Constraint

Before, we only considered carrying capacity when modeling population equations and graphing their patterns. We mentioned about the predator-prey impact on a population, but now, we will consider this impact into our model. The differential equation, with the addition of the predator-prey impact function $p(x)$ is now

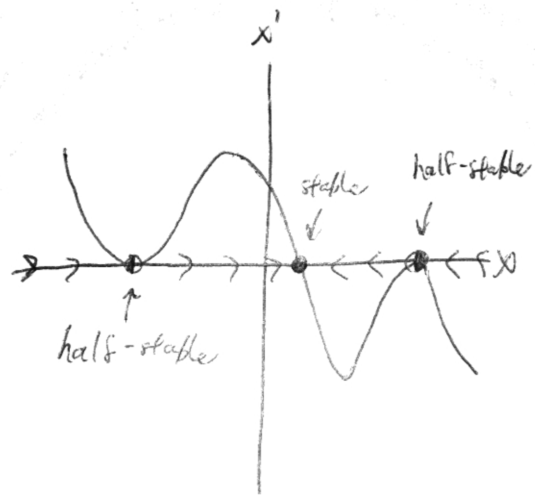

Figure 3: A phase portrait demonstrating a trajectory having two half-stable critical points and one fully stable point

$$\frac{dx}{dt} = \alpha x \left(1 - \frac{x}{\kappa}\right) - p(x) \tag{3}$$

Now we need to know what $p(x)$ is functionally. In *Quanlitative analysis of insect outbreak systems: the spruce budworm and forest*, by D. Ludwig, D. D. Jones, and C. S. Holling [2], the three authors formulated $p(x)$ to be $\frac{Bx^2}{A+x^2}$, where A is the point where predation is the greatest and B is the equilibrium point (the point where predation simmers down to the value given by B). Now we have the equation

$$\frac{dx}{dt} = \alpha x \left(1 - \frac{x}{\kappa}\right) - \frac{Bx^2}{A + x^2} \tag{4}$$

In *Equation 4*, A and B are parameters we added. Unfortunately, the addition of these parameters makes the equation more difficult to graph. In this specific case, it is possible to transform the equation in such a way that A and B disappear.

Here, we introduce arbitrary constants ξ and η (imagine these constants as measurement units, so changing x, t, α or κ will not affect the model), so now we can set

$$\tilde{x} = \xi x, \tilde{t} = \eta t, \tilde{\alpha} = \frac{\alpha}{\eta}, \tilde{\kappa} = \kappa\xi.$$

Setting $\xi = \frac{1}{A}$ and $\eta = \frac{B}{A}$ will get us the desired equation:

$$\frac{d\tilde{x}}{d\tilde{t}} = \tilde{\alpha}\tilde{x}\left(1 - \frac{\tilde{x}}{\tilde{\kappa}}\right) - \frac{\tilde{x}^2}{1+\tilde{x}^2}$$

Since the tildes above x, t, α, and κ are used to distinguish these variables from the original ones, we can substitute the original variables back into *Equation 4* to get

$$\frac{dx}{dt} = \alpha x\left(1 - \frac{x}{\kappa}\right) - \frac{x^2}{1+x^2} \tag{5}$$

which is a more appropriate equation because the number of parameters (α and κ) is less than or equal to the number of variables (x and t) in the equation.

3 Tangent Planes of Implicit Equations

3.1 Parameterizing Equations to Obtain Tangent Planes

Recalling from calculus, the equation of a tangent plane at a point (x_0, y_0, z_0) to a functional surface characterized by a function $f(x, y)$ is the following:

$$0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + (z - z_0) \tag{6}$$

where $f_x(x, y)$ is the partial derivative of f with respect to x and $f_y(x, y)$ is the partial derivative of f with respect to y. Unfortunately, this formula can only apply to a surface whose equation is characterized by a function $z = f(x, y)$ (an equation such that for each $x, y \in \mathbb{R}$, there exists no more than one value of z). If we are given a surface in $\mathbb{R}^3$ such that it cannot be expressed as a function, then the best method of expressing the surface is by using parametric equations $\{x(u, v), y(u, v), z(u, v)\}$, so that z does not need to depend on x and y.

We will look at an equation $\Phi(x, y, z) = 0$, such that it is not possible to obtain a single function z. For example, we have an equation

$$x + 3yz - z^3 = 0 \tag{7}$$

Here, it is not possible to solve for z without having a variable z on the other side of the equation. Although we could solve *Equation 7* for x instead of for z, finding the equation of the tangent plane in terms of z would require some tedious work; one needs to find the partial derivatives and apply it to the tangent plane equation by swapping the variables around in *Equation 6*. Instead of solving for one of the three variables and treat the equation as a function of the other two variables, we will parametrize it.

Because we can solve *Equation 7* for $x = f(y, z)$ (i.e. $x = z^3 - 3yz$), we could set $y = u$ and $z = v$ to get $x = v^3 - 3uv$. The parametrization of *Equation 7* (in vector form) is as follows:

$$R(u, v) = \left\langle v^3 - 3uv, u, v \right\rangle \tag{8}$$

Note that the tangent plane equation $a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$ is characterized by its normal vector $\langle a, b, c\rangle$, which is orthogonal to the tangent plane itself. Now because a surface's partial derivative vectors at a point (x_0, y_0, z_0) are on the tangent plane of the point, then the normal vector can be characterized by the cross product of the partial derivative vectors because the normal vector and the resulting cross product vector are parallel to each other (see *Figure 4*).

The partial derivatives of $R(u, v)$ with respect to the variables u and v are $R_u = \langle -3v, 1, 0\rangle$ and $R_v = \langle 3v^2 - 3u, 0, 1\rangle$. Taking the cross product of R_u and R_v, we get $\langle 1, 3z, 3y - 3z^2\rangle$. Now our normal vector of the tangent plane at the point (x_0, y_0, z_0) is $\langle 1, 3z_0, 3y_0 - 3z_0^2\rangle$, which means that an equation of the tangent plane is the following:

$$(x - x_0) + 3z_0(y - y_0) + (3y_0 - 3z_0^2)(z - z_0) = 0 \tag{9}$$

We are interested in when the tangent plane cannot solved for z. For the surface described by *Equation 7*, the only case when the equation cannot be solved for z (this means the tangent plane could be vertical) is when $3y_0 - 3z_0^2 = 0$, or, more specifically, when $y_0 = z_0^2$.

3.2 Taking the Gradient of $\Phi(x, y, z)$

A shorter, simpler way of finding the equation of a tangent plane, given a surface $\Phi(x, y, z) = 0$, is by forming the gradient vector

$$\nabla\Phi(x, y, z) = \langle \Phi_x(x, y, z), \Phi_y(x, y, z), \Phi_z(x, y, z)\rangle .$$

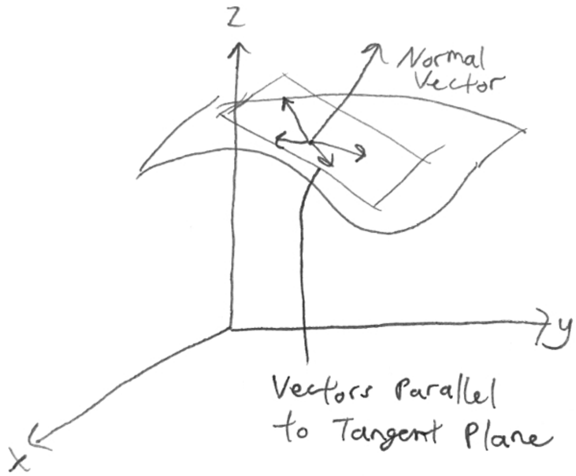

Figure 4: The normal vector is orthogonal to the vectors the tangent plane contains.

The gradient vector is orthogonal to any vector in the tangent plane of a point (x_0, y_0, z_0), so by taking the equation of the tangent plane, we can show that the dot product of $\nabla\Phi$ and the vector $\langle x - x_0, y - y_0, z - z_0\rangle$ is indeed 0 and thus $\nabla\Phi(x_0, y_0, z_0)$ is normal to the tangent plane.

Say that a plane is tangent to Φ at some point (x_0, y_0, z_0). We take a point (x, y, z) on the tangent plane, so we can formulate a vector parallel to the tangent plane $\langle x - x_0, y - y_0, z - z_0\rangle$. Taking the dot product of $\nabla\Phi$ and the vector parallel to the plane, we get

$$\langle \Phi_x(x_0, y_0, z_0), \Phi_y(x_0, y_0, z_0), \Phi_z(x_0, y_0, z_0)\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle$$

$$= \Phi_x(x_0, y_0, z_0)(x - x_0) + \Phi_y(x_0, y_0, z_0)(y - y_0) + \Phi_z(x_0, y_0, z_0)(z - z_0)$$

which is equal to 0, as the dot product above follows the same pattern as the tangent plane equation. Therefore,

$$\langle \Phi_x(x_0, y_0, z_0), \Phi_y(x_0, y_0, z_0), \Phi_z(x_0, y_0, z_0)\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0.$$

and the gradient vector is normal to the tangent plane.

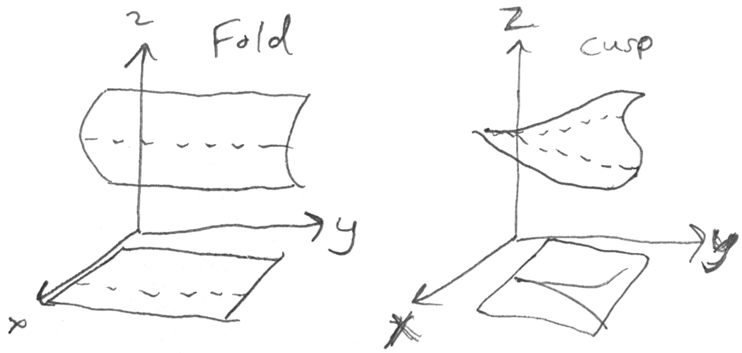

Figure 5:

In the case of *Equation 7*, the gradient vector is $\langle 1, 3z, 3y - 3z^2 \rangle$, which is also normal to the tangent plane at (x_0, y_0, z_0), as we proved earlier. Therefore, the equation of the tangent plane is

$$(x - x_0) + 3z_0(y - y_0) + (3y_0 - 3z_0^2)(z - z_0) = 0$$

which is the same as the one in *Equation 9*.

4 Identifying Folds and Cusps of Surface Projections

In catastrophe theory, given a surface, we are interested in the curves or collection of points that make the tangent planes vertical. Vertical planes have a normal vector whose z-component is equal to 0. In addition, we are also interested in the non-differentiable points of the surface (and thus no tangent plane exists at any of these points). These points cause the surface not to be smooth. For a point to be differentiable at a point in three-dimensional space, all three partial derivatives must be defined at that point.

4.1 Projecting $\Phi(x, y, z)$ on the xy-plane

Projecting a surface on the xy-plane can be useful for identifying points of vertical tangency and non-differentiable points. In Michel Demazure's

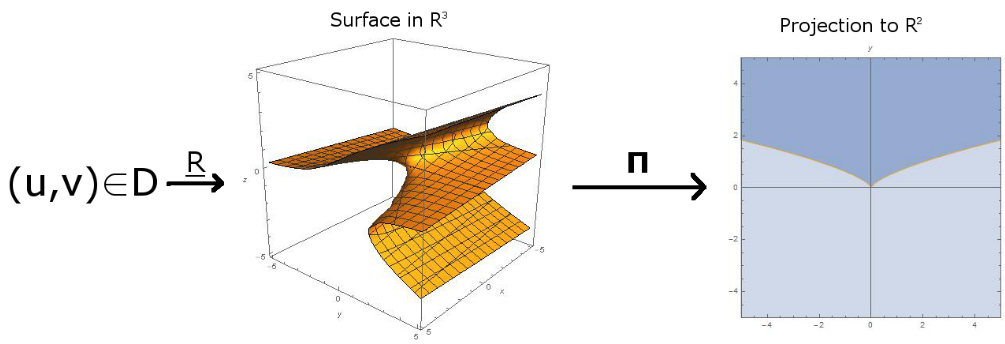

Figure 6: Projection of *Equation 7*. Note that the boundary on the projection is a fold.

Bifurcations and Catastrophes [3], he defines the fold of a surface (or its projection) as the curve on the surface such that all its planes tangent to its points are vertical. Here, $\Phi_z(x, y, z) = 0$, just as described earlier. In addition, he defines the cusp of a surface (or its projection) as the part of the fold such that a particle moving on the fold is moving directly upward or downward. Cusps are characterized by the surface's second partial derivative, both with respect to z, equaling zero ($\Phi_{zz}(x, y, z) = 0$). When projecting a surface with a cusp, the cusp appears as shown in part (b) of *Figure 5*.

To project a surface $\Phi(x, y, z) = 0$ on the xy-plane, first we need to parametrize the surface in terms of two variables. Let's call our variables u and v in a subset of $\mathbb{R}^2$. We define each of the x, y, z variables in $\mathbb{R}^3$ as functions of u and v, which would project an area, characterized by u and v, into three-dimensional space as a surface. To project the surface onto the xy-plane, all we need to do is simply remove the z component of the parametrized surface, so we now have a parametrized area in two-dimensional space. *Figure 6* illustrates the process of the projection, using the surface characterized by *Equation 7*.

The projection Φ on to the xy-plane can be defined as the composition of the projection process Π with the parametrization $\underline{R}$. In other words,

$$\Phi := \Pi \circ \underline{R} : D \subseteq \mathbb{R}^2_{u,v} \to \mathbb{R}^2_{x,y}$$

where Φ is a function of u and v.

4.2 The Determinant of the Jacobian of $\Phi(u, v)$

According to J.W. Bruce and P.J. Giblin in their book *Curves and Singularities* [4], we can easily find the "bad" points of a projection (the points that result in a vertical tangent plane or no tangent plane at all) by taking the determinant of the Jacobian of the projection.
Let $\Phi(u, v) := \langle X(u, v), Y(u, v)\rangle$, the Jacobian of Φ is defined as

$$D\Phi(u, v) := \begin{bmatrix} \frac{\partial X(u,v)}{\partial u} & \frac{\partial X(u,v)}{\partial v} \\ \frac{\partial Y(u,v)}{\partial u} & \frac{\partial Y(u,v)}{\partial v} \end{bmatrix} \tag{10}$$

If we choose a value u and v, we can determine if the surface behaves well (i.e. they are not "bad" values) for those values, by finding the rank of the Jacobian matrix. If $\text{rank}(D\Phi) = 2$, then the surface behaves well for those values. If $\text{rank}(D\Phi) < 2$ or if the rank does not exist, then the values are "bad." The shortest method of determining if the rank of a square matrix is the same as its dimension is by taking the determinant of the matrix. In this case, the determinant of the Jacobian of $\Phi(u, v)$ is shown below:

$$\det(D\Phi(u, v)) = \begin{vmatrix} \frac{\partial X(u,v)}{\partial u} & \frac{\partial X(u,v)}{\partial v} \\ \frac{\partial Y(u,v)}{\partial u} & \frac{\partial Y(u,v)}{\partial v} \end{vmatrix} = \frac{\partial X(u, v)}{\partial u} \cdot \frac{\partial Y(u, v)}{\partial v} - \frac{\partial X(u, v)}{\partial v} \cdot \frac{\partial Y(u, v)}{\partial u} \tag{11}$$

If the determinant of $D\Phi$ is nonzero, then the rank is 2. If the determinant of $D\Phi$ is equal to zero, then the rank is less than 2. If the determinant does not exist, then the rank of the matrix does not exist. The last case can only happen if one of the values in the matrix does not exist.

Let's use the surface in *Equation 7* as our example. Since we know the parametrization of the surface is expressed in *Equation 8*, we can remove the third coordinate to obtain the equation of the projection on to the xy-plane. Therefore, our equation is now $\Phi(u, v) = \langle v^3 - 3uv, u\rangle$, which is shown in *Figure 6*. Taking the determinant of the Jacobian of Φ, we get $-3v^2 + 3u$, which, after reverse substitution (y for u and z for v), is the same as the z-coordinate of the vector, normal to its tangent plane at point (x, y, z).

Recalling from earlier, again, if the z-coordinate of the normal vector of the tangent plane is 0, then the tangent plane is vertical. Therefore, we verified that if we set the determinant of the Jacobian of $\Phi(u, v)$ equal to

0, then all the values of u and v that satisfy this condition are values that result in a vertical tangent plane.

5 Applying the Technique of Projections to Our Population Model

Our enhanced population model is shown in *Equation 5*. The points we are interested in are how each of the two components $\phi_{\alpha,\kappa}(x)$ and $\psi(x)$ interact when $\frac{dx}{dt} = 0$. Given the condition of $\frac{dx}{dt}$,

$$\alpha x \left(1 - \frac{x}{\kappa}\right) = \frac{x^2}{1 + x^2}$$

We can divide x on both sides to simplify the above function to get

$$\alpha \left(1 - \frac{x}{\kappa}\right) = \frac{x}{1 + x^2}$$

Now we can set each side of the equation to functions $\phi_{\alpha,\kappa}(x)$ and $\psi(x)$ respectively:

$$\phi_{\alpha,\kappa}(x) = \alpha \left(1 - \frac{x}{\kappa}\right), \psi(x) = \frac{x}{1 + x^2}$$

The constraints of α and κ are to be given in realistic terms. Therefore, $\alpha, \kappa > 0$. The graphs of $\phi_{1/2,10}(x)$ and $\psi(x)$ are shown in *Figure 7*.

ψ is a function of one variable with no parameters, so the graph's equation does not change in form. On the other hand, ϕ is a function with two parameters, so because the function has more than one parameter, then the graph's equation changes with respect to each of the parameters. As a result, the number of intersections of the two graphs changes.

For all values of α and κ, the two graphs intersect each other at least once. We can show that this is the case by using the intermediate value theorem. The theorem states that given an interval $[a, b]$ such that a function f is continuous on $[a, b]$, then if we have a value d such that $f(a) \leq d \leq f(b)$ or $f(a) \geq d \geq f(b)$, then there exists a $c \in (a, b)$ such that $f(c) = d$. In the case of our population model, $\phi_{\alpha,\kappa}(x)$ always intersects the x-axis at $(\kappa, 0)$, but $\psi(x)$ never intersects the same axis. This means that the difference $\psi(x) - \phi_{\alpha,\kappa}(x)$ is guaranteed to be less than 0 for any value of x larger than κ. At the same time, because $\phi_{\alpha,\kappa}$ intersects the y-axis at the point $(0, \alpha)$ where

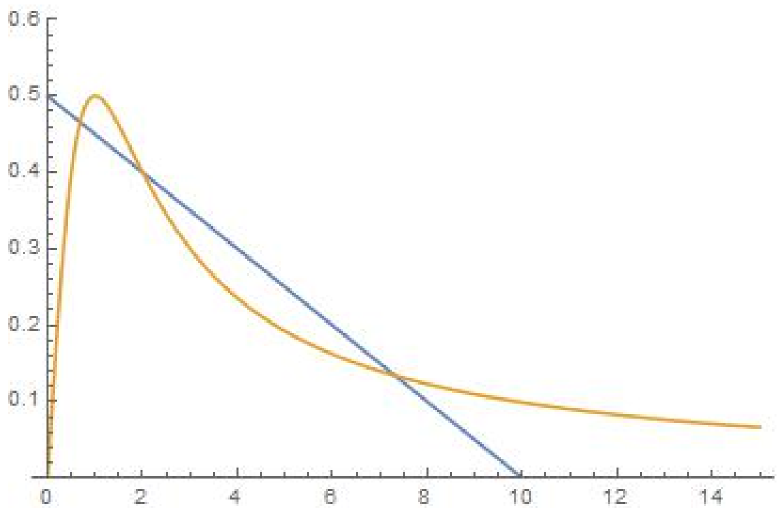

Figure 7: $\phi_{1/2,10}(x)$ is in blue and $\psi(x)$ is in orange.

$\alpha > 0$, then $\phi_{\alpha,\kappa}(0) > 0$. Additionally, $\psi(0) = 0$, so $\psi(0) - \phi_{\alpha,\kappa}(0) > 0$. Because both ϕ and ψ are continuous on the set of all non-negative real numbers, then we can use the intermediate value theorem to say that for some value $n \in (0, \kappa)$, $\psi(n) - \phi_{\alpha,\kappa}(n) = 0$ and thus ϕ intersects ψ at least once.

There can be as many as three intersections between the two graphs (this is because $\phi(x)$ is a line and $\psi(x)$ is a curve with one inflection point, according to *Figure 7*). Even though the two graphs can intersect each other once, twice, or three times, the interesting part of the intersection of ϕ and ψ is that when they intersect twice, $\phi_{\alpha,\kappa}(x)$ is tangent to $\psi(x)$ at one of the intersections. However, when $\phi_{\alpha,\kappa}(x)$ is tangent to $\psi(x)$ at its inflection point, then the two graphs intersect each other only once (at that point).

The goal now is to find all the values of α and κ that make $\phi_{\alpha,\kappa}(x)$ tangent to $\psi(x)$:

First, let's create a surface satisfying the equation $\phi_{\alpha,\kappa}(x) - \psi(x) = 0$ where Σ defines the surface as a function in the (α, κ, x) coordinate system. In other words,

$$\Sigma(\alpha, \kappa, x) := \alpha\left(1 - \frac{x}{\kappa}\right) - \frac{x}{1 + x^2} = 0$$

In order to project the surface on to the $\alpha\kappa$-plane, we need to parametrize the surface. To do this, we will solve for α and have our parameters be $\kappa = u$

and $x = v$. Our parametrized surface equation is now

$$\Phi(u,v) = \left\langle \frac{v}{(1+v^2)(1-v/u)}, u, v \right\rangle$$

which can be simplified to

$$\Phi(u,v) = \left\langle \frac{uv}{(1+v^2)(u-v)}, u, v \right\rangle \tag{12}$$

by eliminating the fraction v/u. Now we can project *Equation 12* on to the $\alpha\kappa$-plane by eliminating the third component $x = v$. The equation is now

$$\Phi(u,v) = \langle A(u,v), K(u,v)\rangle = \left\langle \frac{uv}{(1+v^2)(u-v)}, u \right\rangle \tag{13}$$

We wrote at the beginning of this section that we want $\frac{dx}{dt}$ to equal 0. Equivalently, to determine when the x component of the normal vector to the tangent plane is 0, we determine when $\det(D\Phi(u,v)) = 0$:

$$\det(D\Phi(u,v)) = \begin{vmatrix} \frac{-v^2}{(1+v^2)(u-v)^2} & \frac{u^2-u^2v^2+2uv^3}{(1+v^2)^2(u-v)^2} \\ 1 & 0 \end{vmatrix} = \frac{-u^2+u^2v^2-2uv^3}{(1+v^2)^2(u-v)^2} = 0$$

The determinant does not exist when $u - v = 0$ or, furthermore, when $u = v$. Otherwise, the determinant is equal to 0 if $-u^2 + u^2v^2 - 2uv^3 = 0$. Since $u = \kappa > 0$, we can divide u on both sides to get $-u + uv^2 - 2v^3 = 0$. Solving for u, we get $u = \frac{2v^3}{v^2-1}$, which we can treat as a function of v, namely $U(v) := \frac{2v^3}{v^2-1}$. We can substitute $U(v)$ for u in *Equation 13* to get $\Phi(U(v), v)$.

$$\Phi(U(v),v) = \langle A(U(v),v), K(U(v),v)\rangle = ... = \left\langle \frac{2v^3}{(1+v^2)^2}, \frac{2v^3}{v^2-1} \right\rangle =: \mathbf{r}(v)$$

$\mathbf{r}(v)$, where $v > 1$, is the parametric equation which characterizes the fold of the surface Σ, when projected on to the $\alpha\kappa$-plane. A graph of the curve from $1 < v < 5$ is shown in *Figure 8*:

Observing the graph of $\mathbf{r}(v)$, the curve appears to not be differentiable at one spot. We must find all the values of v where $\mathbf{r}$ is not differentiable. If a particle is traveling along the curve, we can calculate its speed and direction by taking the derivative of $\mathbf{r}(v)$:

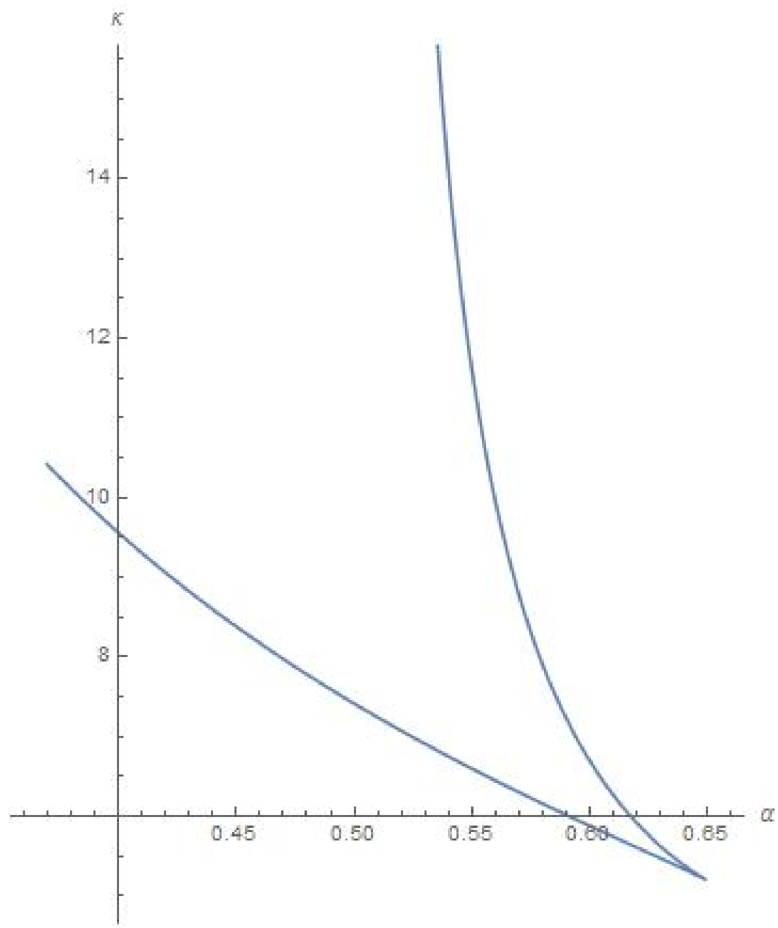

Figure 8: $\mathbf{r}(v) = \left\langle \frac{2v^3}{(1+v^2)^2}, \frac{2v^3}{v^2-1} \right\rangle$

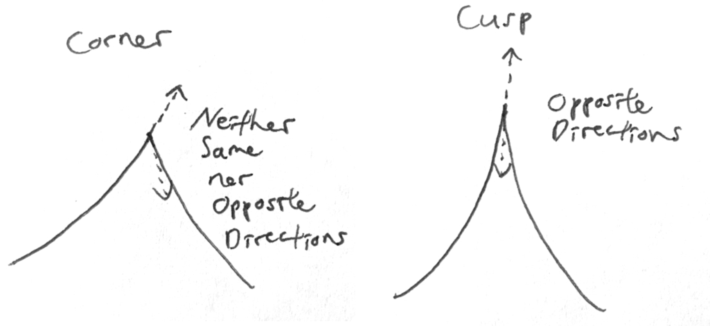

Figure 9: The dashed arrows show the direction of the unit vector before and after the non-differentiable points.

$$\mathbf{r}'(v) = \left\langle \frac{2v^2(3-v^2)}{(1+v^2)^3}, \frac{2v^2(v^2-3)}{(v^2-1)^2} \right\rangle \tag{14}$$

In general, continuous functions which are not differentiable everywhere have corners or cusps. Earlier, we described what cusps are, in comparison to folds. But now, with a new term "corner," we will compare cusps to corners by first geometrically defining a corner:

A corner is a sharp turn on a curve such that the limits of two unit vectors approaching the corner from opposite directions are unequal, but are not opposite. Geometrically speaking, a cusp is like a corner, but has opposite limits. *Figure 9* shows a corner and a cusp with limits of unit vectors on each.

Now that we know $\mathbf{r}'(v)$, to remove the speed property of the vector equation, we need to convert the vector equation to a unit vector equation by dividing $\mathbf{r}'(v)$ by its magnitude:

$$\begin{aligned} ||\mathbf{r}'(v)|| &= \sqrt{\left(\frac{2v^2(3-v^2)}{(1+v^2)^3}\right)^2 + \left(\frac{2v^2(v^2-3)}{(v^2-1)^2}\right)^2} \\ &= \sqrt{\left(\frac{-2v^2(v^2-3)}{(1+v^2)^3}\right)^2 + \left(\frac{2v^2(v^2-3)}{(v^2-1)^2}\right)^2} \end{aligned}$$

$$= \sqrt{(2v^2(v^2-3))^2 \left(\frac{1}{(1+v^2)^6} + \frac{1}{(v^2-1)^4}\right)}$$
$$= \pm 2v^2(v^2-3)\sqrt{\frac{1}{(1+v^2)^6} + \frac{1}{(v^2-1)^4}}$$

For the unit vector equation to be undefined, we must find all v such that $||\mathbf{r}'(v)|| = 0$. $v = \sqrt{3}$ is the only solution and will therefore be the point where the corner or cusp occurs on the curve. By dividing $\mathbf{r}'(v)$ by its magnitude, we get

$$\frac{\mathbf{r}'(v)}{||\mathbf{r}'(v)||} = \left\langle \frac{\mp 1}{\sqrt{1+\frac{(1+v^2)^6}{(v^2-1)^4}}}, \frac{\pm 1}{\sqrt{\frac{(v^2-1)^4}{(1+v^2)^6}+1}} \right\rangle, v \neq \sqrt{3}$$

And lastly, by taking the limit of the unit vector equation at $v = \sqrt{3}$ from both sides, we get

$$\left\langle \frac{-1}{\sqrt{257}}, \frac{16}{\sqrt{257}} \right\rangle \text{ and } \left\langle \frac{1}{\sqrt{257}}, \frac{-16}{\sqrt{257}} \right\rangle$$

which point into opposite directions. Therefore, $\mathbf{r}(v)$ has a cusp, as shown in *Figure 10.*

Before we conclude this study, we would like to know the values of α and κ at the cusp:

$$\alpha = \frac{3\sqrt{3}}{8} \approx 0.6495$$
$$\kappa = 3\sqrt{3} \approx 5.1962$$

6 Summary

In population modeling, like in all mathematical models, constraints make the model more complex, yet more applicable to real-life scenarios. For example, although the population of a species, including ours, can grow indefinitely, the carrying capacity constraint tells us that a habitat can only accommodate so many organisms at once. Additionally, the predation constraint limits the presence of one species for the benefit of another species.

Throughout the Fall 2016 semester, we were interested in the measurements that characterized a population, namely the growth α and carrying

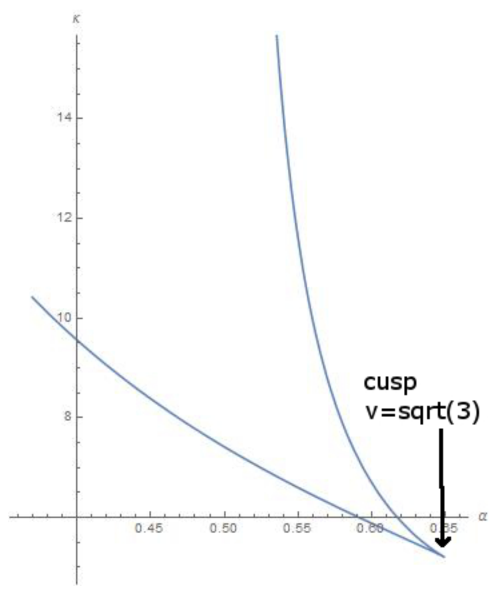

Figure 10: $\mathbf{r}(v)$ with cusp identified

capacity κ. Given that the size of the population did not change over time, we were curious how and in what manner the characterization measurements changed. We noticed that there was a sudden change in population measurement once the growth of the population reached to 0.6495 organisms per given time and the carrying capacity of the habitat (assuming that it could change) was 5.1962 units of organisms. These values characterize a cusp, which represents a sudden change in tendency, in the graph of $\mathbf{r}(t)$ on the $\alpha\kappa$-plane. Somehow, these values of $\alpha = 0.6495$ and $\kappa = 5.1962$ also characterize the line $\phi_{0.6495,5.1962}(x)$ tangent to $\psi(x)$ at its inflection point. An interesting question would be if there is a connection between the cusp point of a graph and the inflection point of the corresponding graph.

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