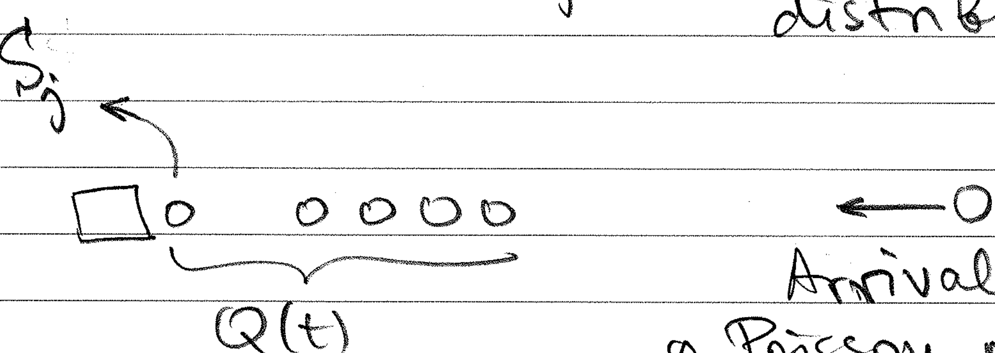
M(λ)/G/1

Inter-arrival times $\tau_j \sim \text{Exp}(\lambda)$.

Service times S_j have some known distribution.



Arrivals form a Poisson process with rate λ

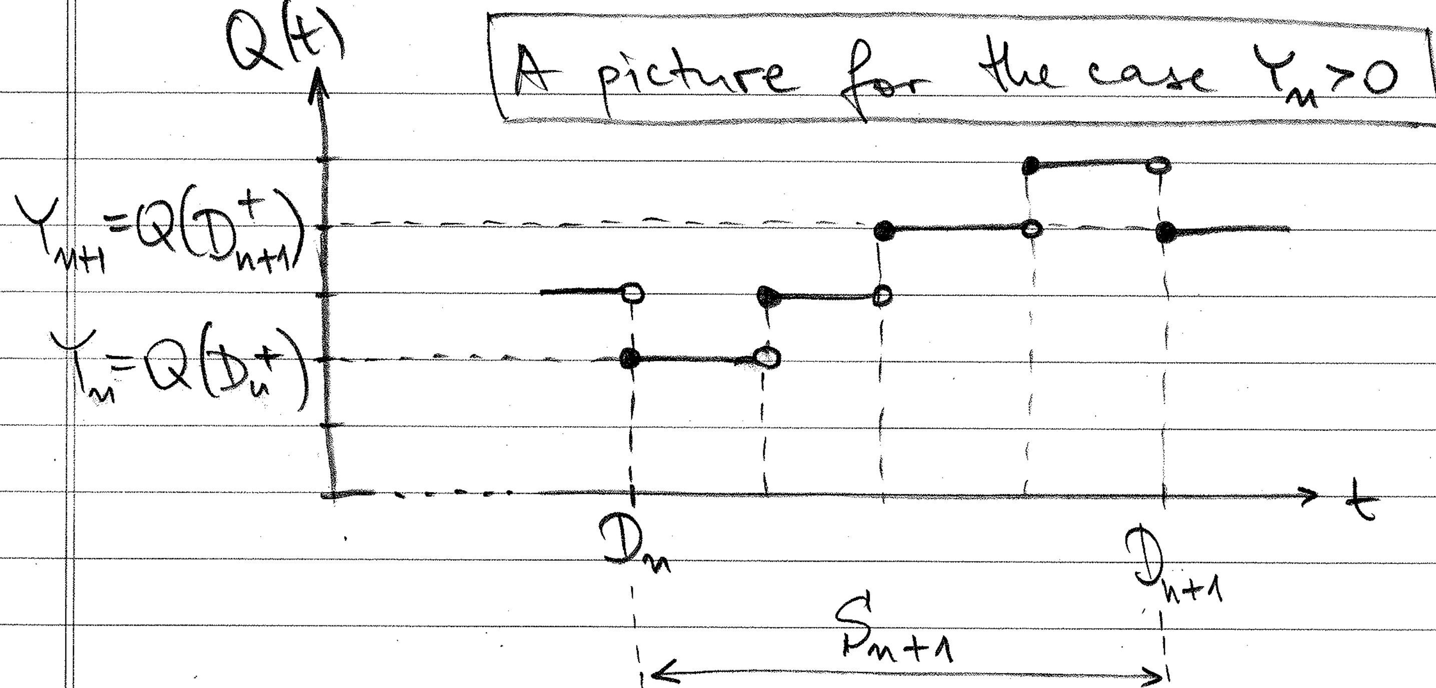
There is a discrete-time Markov chain

$$Y = \{Y_n : n \geq 1\}$$

embedded in $Q(t)$. Let

- $D_n :=$ time of departure of the n^{th} customer from the system
- $Y_n := Q(D_n^+)$

  $=$ # of customers which the n^{th} customer leaves behind after he/she leaves



- U_n := # of customers arriving during the service time (of length S_{n+1}) of the $(n+1)$st customer.

  In the picture above, $U_n = 3$.

We have:

- if $Y_n = Q(D_n^+) > 0$,

$$Y_{n+1} = Y_n + U_n - 1;$$

- if $Y_n = Q(D_n^+) = 0$,

$$Y_{n+1} = U_n.$$

Note that $\{Y_n\}$ is a Markov chain!

Let p_{ij} be the transition probability matrix of the discrete MC $\{Y_n\}$.

Computing p_{ij} for $i \geq 1$, $j \geq i-1$:

$$p_{ij} = \mathbb{P}(Y_{n+1} = j \mid Y_n = i)$$

$$= \mathbb{P}(j = i + U_n - 1)$$

$$\boxed{\text{because } Y_n \geq 1, \text{ so that } Y_{n+1} = Y_n + U_n - 1}$$

$$= \mathbb{P}(U_n = j - i + 1)$$

$$= \mathbb{E}\left[\mathbb{P}(U_n = j-i+1 \mid S_{n+1})\right]$$

$$= \int \mathbb{P}(U_n = j-i+1 \mid S_{n+1} = s)\, f_S(s)\, ds$$

$$= \int \mathbb{P}\begin{pmatrix} j-i+1 \text{ events} \\ \text{of the Poisson arrival} \\ \text{process of rate } \lambda \\ \text{during time interval} \\ \text{of length } s \end{pmatrix} f_S(s)\, ds$$

$$= \int \frac{(\lambda s)^{j-i+1}}{(j-i+1)!} e^{-\lambda s} f_S(s)\, ds$$

$$= \mathbb{E}\left[\frac{(\lambda S)^{j-i+1}}{(j-i+1)!} e^{-\lambda S}\right]$$

Working similarly, one can obtain that the 1-step transition probability matrix of the discrete MC $\{Y_n\}$ is

$$\underline{P} = \begin{pmatrix} \delta_0 & \delta_1 & \delta_2 & \delta_3 & \delta_4 & \cdots \\ \delta_0 & \delta_1 & \delta_2 & \delta_3 & \delta_4 & \cdots \\ 0 & \delta_0 & \delta_1 & \delta_2 & \delta_3 & \cdots \\ 0 & 0 & \delta_0 & \delta_1 & \delta_2 & \cdots \\ 0 & 0 & 0 & \delta_0 & \delta_1 & \cdots \\ 0 & 0 & 0 & 0 & \delta_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where

$$\delta_j = \mathbb{E}\left[\frac{(\lambda S)^j}{j!} e^{-\lambda S}\right].$$