Math 240: Elementary Differential Equations

Exam 1: Summer 2010

June 18, 2010

Name.......................

Instructor.................

This is a closed book exam. You can use a calculator and 8.5" × 11" sheet of hand written note (both sides). Detail work must be shown for the full credit.

(10)Problem.1. Find all the solution to the differential equation

$$\frac{dy}{dx} = \frac{xy}{x^2+1}$$

Separable! 2

- $y(x) \equiv 0$ is a solution.
- $y(x) \not\equiv 0$ then $\frac{1}{y}\,dy = \frac{x}{x^2+1}\,dx.$ 2

$$\int \frac{1}{y}\,dy = \ln|y| + c_1 = \int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{dx^2}{1+x^2}$$

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$$= \frac{1}{2}\ln(x^2+1) + c_2.$$

2

Thus

$$\ln|y| = \frac{1}{2}\ln(1+x^2) + C$$

$$|y| = e^{\ln|y|} = e^{\frac{1}{2}\ln(1+x^2) + C}$$

$$= e^{\frac{1}{2}\ln(1+x^2)}\, e^{C}$$

$$= (1+x^2)^{\frac{1}{2}} \cdot D, \qquad D := e^{C}.$$

Solution

$$y = (1+x^2)^{\frac{1}{2}}\, C$$

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(10)Problem 2. Solve the initial value problem :

$$\underbrace{3x^2 \tan(y)}_{M} + 1 \underbrace{\left(x^3 \sec^2(y) - 1\right)}_{N} \frac{dy}{dx} = 0, y(0) = 0$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 3x^2 \sec^2 y. \\ \frac{\partial N}{\partial x} &= 3x^2 \sec^2 y \end{aligned}\right\} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \underline{\text{exact}}.$$

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$$F(x,y) = \int M\,dx = \int 3x^2 \tan(y)\,dx = x^3 \tan(y) + C_1(y)$$

$$F(x,y) = \int N\,dy = \int \left[x^3 \sec^2(y) - 1\right] dy = x^3 \tan(y) - y + C_2(x)$$

Thus

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$$F(x,y) = x^3 \tan(y) - y$$

<u>General solution:</u>

$$x^3 \tan(y) - y = C$$

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$$y(0) = 0 \Rightarrow C = 0.$$

<u>Solution</u> $\boxed{x^3 \tan(y) - y = 0}$

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(10) Problem 3. Find all solutions to the equation

$$e^x \frac{dy}{dx} + 2e^x y = 1.$$

$$\frac{dy}{dx} + 2y = e^{-x}$$ linear! 2

$$\mu(x) := e^{\int 2\,dx} = e^{2x}.$$ 2 (Integrating factor)

$$\frac{d}{dx}\left(y(x)\, e^{2x}\right) = e^{-x} e^{2x} = e^{x}$$ 2

$$y(x)\, e^{2x} = \int e^{x}\, dx = e^{x} + C$$ 2

$$y(x) = e^{-2x}\left(e^{x} + C\right) = e^{-x} + Ce^{-2x}$$

So $$\boxed{y(x) = e^{-x} + Ce^{-2x}}$$ 2

(10)Problem 4. Find all solutions to the equation

$$\frac{dy}{dx} - 4y = -xy^3$$

Bernoulli with $n=3$ 1

$$y = v^{\frac{1}{1-3}} = v^{\frac{1}{-2}} \quad 1 \quad \Rightarrow y^2 = v^{-1} \Rightarrow v = y^{-2}.$$

$$\frac{dv}{dx} - 4(1-3)v = -x(1-3)$$

$$\frac{dv}{dx} + 8v = 2x \quad 2 \qquad \mu(x) = e^{\int 8dx} = e^{8x}$$

$$\frac{d}{dx}\left(v(x)e^{8x}\right) = e^{8x}\cdot 2x$$

$$v(x)e^{8x} = \int e^{8x}2x\,dx = \frac{1}{8}\int 2x\,de^{8x}$$

by parts

$$= \frac{1}{8}\left(2xe^{8x} - \int e^{8x}d(2x)\right) \quad 2$$

$$= \frac{1}{8}\left(2xe^{8x} - 2\int e^{8x}dx\right)$$

$$= \frac{1}{8}\left(2xe^{8x} - 2\cdot\frac{1}{8}e^{8x}\right) + C$$

So

$$v(x) = e^{-8x}\left(\frac{x}{4}e^{8x} - \frac{1}{32}e^{8x} + C\right)$$

$$= \frac{x}{4} - \frac{1}{32} + Ce^{-8x} \quad 2$$

Thus

$$y(x) = \frac{1}{\sqrt{v(x)}} = \frac{1}{\sqrt{\frac{x}{4} - \frac{1}{32} + Ce^{-8x}}} \quad 2$$

(20)Problem 5. Suppose

$$\frac{dP}{dt} = -P^3 + 10P^2 - 21P.$$

(a) Find the equilibrium points of the equation. Sketch the several solution curves indicating the threshold. Also classify the equilibrium points as stable or unstable.

$$f(P) := -P^3 + 10P^2 - 21P = -P(P^2 - 10P + 21)$$

$$= -P(P-3)(P-7).$$ Eq. pts are $0, 3, 7$

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P		0		3		7	∞
$f(P)$	$+$	0	$-$	0	$+$	0	$-$

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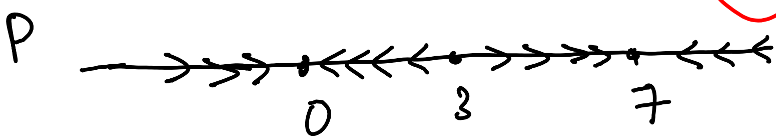

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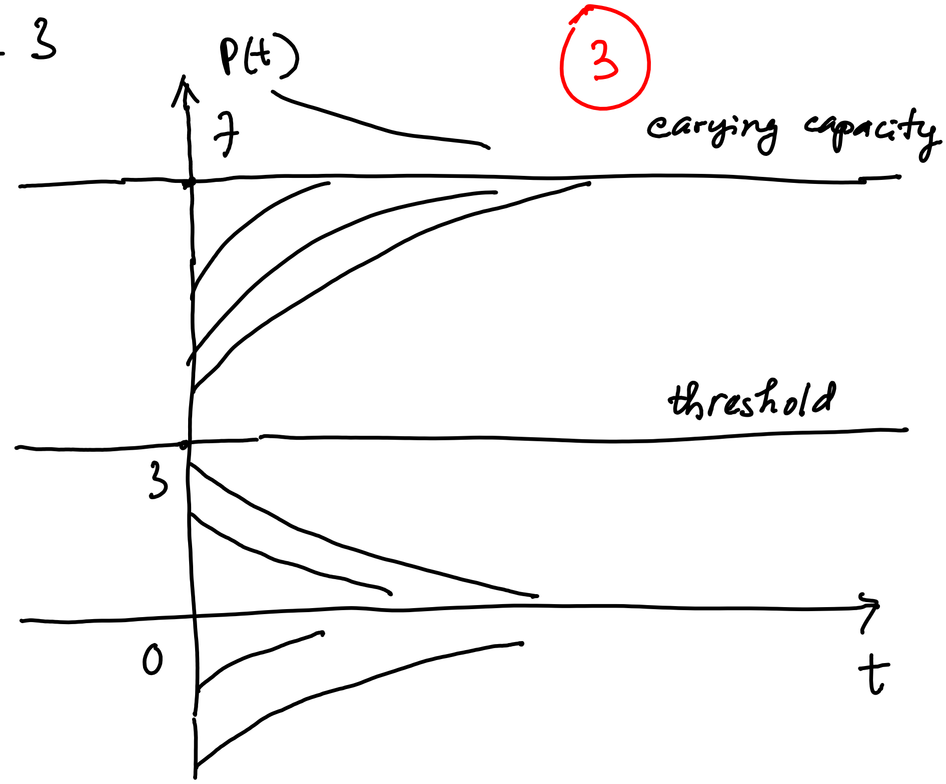
Stable eq. pts. $P=0$, $P=7$.

Unstable eq. pt. $P=3$

3



(b) If $P(0) = 6$ find the limits

$$\lim_{t\to\infty} P(t) \quad \text{and} \quad \lim_{t\to-\infty} P(t).$$

$\lim_{t\to\infty} P(t) = 7$, $\lim_{t\to-\infty} P(t) = 3$.

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(c). Suppose $P(t)$ represents population of fish in a lake in thousands.What is the fate of the population of the fish in the lake if the current population of the fish is 1500?

$P(0) = 1.5$ (thousand) below threshold.

$\lim_{t\to\infty} P(t) = 0$. Extiction !

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(d). How many fish should be in the lake in order to prevent the extinction ?

$P \geqslant$ threshold $= 3$ (thousands)

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(10)**Problem 6 (a).** Express the following in the form of $a + bi$.

$$\frac{2-3i}{4+3i}$$

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$$\frac{2-3i}{4+3i} = \frac{(2-3i)(4-3i)}{(4+3i)(4-3i)} = \frac{8-6i-12i+9i^2}{4^2+3^2}$$

$$= \frac{-1-18i}{25} = \boxed{\frac{-1}{25} + \frac{-18}{25}i}$$

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(b). Express $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ into polar form.

$$|z| = \sqrt{\left(-\tfrac{1}{2}\right)^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2} = \sqrt{\tfrac{1}{4}+\tfrac{3}{4}} = 1.$$

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$$Re(z) = -\frac{1}{2} < 0 \; ; \qquad \arg(z) = \operatorname{atan}\left(\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right) + \pi$$

$$= \pi + \tan(-\sqrt{3}) = \pi - \frac{\pi}{3}$$

$$= \frac{2\pi}{3}$$

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So $\boxed{z = 1 . e^{i\frac{2\pi}{3}}}$

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(10)**Problem 7(a).** Write the following in the form of $A\cos(\omega x - \phi)$

$$Z := 2cos(3x) + 2\sqrt{3}sin(3x).$$

- $Z = Re\left(2e^{i3x}\right) + Re\left(2\sqrt{3}(-i)e^{i3x}\right)$

$$= Re\left(2e^{i3x} + 2\sqrt{3}(-i)e^{i3x}\right) = Re\left(e^{i3x}(2 - 2\sqrt{3}i)\right)$$

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- $Z_1 = 2 - 2\sqrt{3}i$

$|Z_1| = \sqrt{2^2 + (-2\sqrt{3})^2} = \sqrt{4+12} = 4.$

$\arg(Z_1) = \operatorname{atan}\left(\frac{-2\sqrt{3}}{2}\right) = \operatorname{atan}(-\sqrt{3}) = -\frac{\pi}{3}.$

$Z_1 = 4e^{-i\frac{\pi}{3}}.$

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So

$$Z = Re\left(e^{i3x} \cdot 4e^{-i\frac{\pi}{3}}\right) = 4\,Re\left(e^{i(3x - \frac{\pi}{3})}\right)$$

$$\boxed{2\cos(3x) + 2\sqrt{3}\sin(3x) = 4\cos\left(3x - \frac{\pi}{3}\right)}$$

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(b). Express $2\cos(3x - \frac{\pi}{3})$ in the form of $a\cos(\omega x) + b\sin(\omega x)$.

- From part a): $2\cos(3x) + 2\sqrt{3}\sin(3x) = 4\cos\left(3x - \frac{\pi}{3}\right).$

So

$$\boxed{2\cos\left(3x - \frac{\pi}{3}\right) = \cos(3x) + \sqrt{3}\sin(3x)}$$

- Alternatively,

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$$2\cos\left(3x - \frac{\pi}{3}\right) = 2\,Re\left(e^{i(3x - \frac{\pi}{3})}\right) = 2\,Re\left(e^{i3x} \cdot e^{i(-\frac{\pi}{3})}\right)$$

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$$= 2\,Re\left[\left(\cos(3x) + i\sin(3x)\right)\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)\right]$$

$$= 2\left[\cos(3x)\cos\left(-\frac{\pi}{3}\right) - \sin(3x)\sin\left(-\frac{\pi}{3}\right)\right]$$

$$= \cos(3x) + \sqrt{3}\sin(3x).$$

1