CALCULUS 3 FALL 2010

EXAM 2

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REC. INSTR.______________

REC. TIME______________

TO RECEIVE CREDIT YOU MUST SHOW YOUR WORK

(15) 1. Suppose that Q depends upon x and y according to $Q = 5x/(x^2 + y^2)$. X and y are changing with the time t and at a certain instant you know that $x = 1$, $y = 2$, $dx/dt = 2$ and $dy/dt = 4$. Use the chain rule to find dQ/dt at this instant.

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial x}\frac{dx}{dt} + \frac{\partial Q}{\partial y}\frac{dy}{dt} \quad \Big\} \; 3$$

$$= \Bigg[\underbrace{\frac{(x^2+y^2)(5) - (5x)(2x)}{(x^2+y^2)^2}}_{4} \Bigg] \frac{dx}{dt} + \Bigg[\underbrace{\frac{(x^2+y^2)(0) - (5x)(2y)}{(x^2+y^2)^2}}_{4} \Bigg] \frac{dy}{dt}$$

$$= \frac{5(y^2 - x^2)}{(x^2+y^2)^2}\frac{dx}{dt} - \frac{10xy}{(x^2+y^2)^2}\frac{dy}{dt}$$

PUT IN DATA

$$= \frac{(5)(3)}{25}(2) - \frac{(20)}{25}(4) = \frac{30-80}{25} = \frac{-50}{25} = -2 \quad \Big\} \; 4$$

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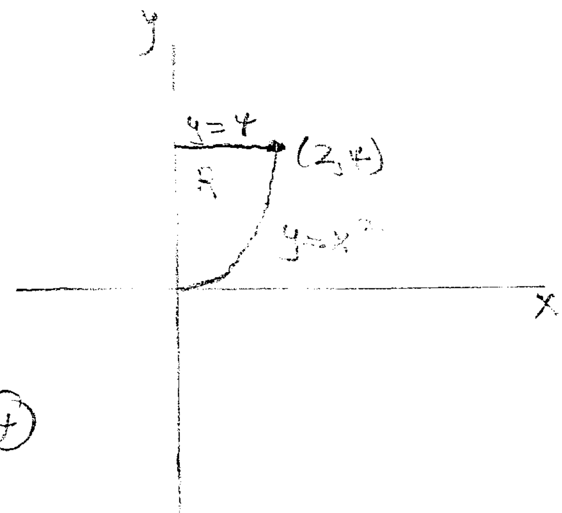
(15) 2. Calculate the volume of the 3D-region which is under $z = 1 + 2xy$ and above the region in the first quadrant of the xy-plane that is enclosed by $y = x^2$, $y = 4$ and $x = 0$.



$$\text{Volume} = \iint_R (1+2xy)\,dA \quad 4$$

$$= \int_0^2 \int_{x^2}^4 (1+2xy)\,dy\,dx \quad 4$$

$$= \int_0^2 (y + xy^2)\Big|_{y=x^2}^{y=4} dx = \int_0^2 [(4+16x)-(x^2+x^5)]\,dx \quad 4$$

$$= 4x + 8x^2 - \frac{x^3}{3} - \frac{x^6}{6}\Big|_0^2 = 8+32-\frac{8}{3}-\frac{32}{3} = 40-\frac{40}{3} = \frac{80}{3} \quad 3$$

OR

$$= \int_0^4 \int_0^{\sqrt{y}} (1+2xy)\,dx\,dy = \int_0^4 (x + x^2 y)\Big|_{x=0}^{x=\sqrt{y}} dy$$

$$= \int_0^4 (y^{\frac{1}{2}} + y^2)\,dy$$

$$= \frac{2}{3} y^{3/2} + \frac{y^3}{3}\Big|_0^4 = \frac{16}{3} + \frac{64}{3} = \frac{80}{3}$$

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(15) 3. Find the equation of the tangent plane to the surface

$$\underbrace{xz^3 + x^3y + y^3z + 7}_{F(x,y,z)} = 0 \quad \text{at the point } (x,y,z) = (1,2,-1) .$$

$$\nabla F = \underbrace{(z^3 + 3x^2y)}_{2}\vec{i} + \underbrace{(x^3 + 3y^2z)}_{2}\vec{j} + \underbrace{(3xz^2 + y^3)}_{2}\vec{k}$$

$$\nabla F(1,2,-1) = (-1+6)\vec{i} + (1-12)\vec{j} + (3+8)\vec{k}$$

$$\left.\nabla F(1,2,-1) = 5\vec{i} - 11\vec{j} + 11\vec{k}\right\} 4$$ is a perp. vector for the tangent plane at $(1,2,-1)$

Thus the equation of the tangent plane at $(1,2,-1)$ is

$$\left.5(x-1) - 11(y-2) + 11(z+1) = 0\right\} 5$$

$$5x - 11y + 11z + 28 = 0$$

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(20) 4. Let $f(x,y) = xe^{(x/y)}$.

a) find the gradient vector field of f

$$\nabla f = \left(e^{x/y} + \frac{x}{y} e^{x/y} \right)\vec{i} - \frac{x^2}{y^2} e^{x/y} \vec{j} \quad \} \; 4$$

b) find the directional derivative of f(x,y) at (x,y) = (2,1) in the direction of the vector **v** = **i** + 2**j**

$$|\vec{v}| = \sqrt{5} \quad \text{so} \quad \vec{u} = \frac{1}{|\vec{v}|}\vec{v} = \frac{1}{\sqrt{5}}\vec{i} + \frac{2}{\sqrt{5}}\vec{j} \quad \} \; 2$$

$$\nabla f(2,1) = e^2(1+2)\vec{i} - e^2 4\vec{j} = 3e^2\vec{i} - 4e^2\vec{j} \quad \} \; 2$$

$$D_u f(2,1) = \nabla f(2,1)\cdot\vec{u} = \frac{3e^2}{\sqrt{5}} - \frac{8e^2}{\sqrt{5}} = \frac{-5e^2}{\sqrt{5}} = -\sqrt{5}\,e^2 \quad \} \; 6$$

c) find the largest directional derivative of f(x,y) at (x,y) = (2,1)

$$|\nabla f(2,1)| = |e^2(3\vec{i} - 4\vec{j})| = 5e^2 \quad \} \; 3$$

d) find a unit vector which points in the direction which gives the largest directional derivative that you found in part c)

$$\frac{\nabla f(2,1)}{|\nabla f(2,1)|} = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j} \quad \} \; 3$$

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(20) 5. Let $f(x,y) = (\sin x)(y^2 - 1)$

a) find all the critical points of $f(x,y)$ which are in the region of the xy-plane that has $0 < x < 2\pi$

b) apply the 2nd partials test to each of the critical points that you found in part a) to determine its nature

$$\frac{\partial f}{\partial x} = (\cos x)(y^2-1) = 0, \quad \frac{\partial f}{\partial y} = \underbrace{(\sin x)(2y) = 0}_{\text{so } y=0 \text{ OR } x=\pi} \quad (4)$$

$x=\pi$ gives $y^2-1=0$

so $y=\pm 1$

$y=0$ gives $\cos x = 0$

so $x = \frac{\pi}{2}, \frac{3\pi}{2}$

Thus CP's: $(x,y) = (\pi,1), (\pi,-1), (\frac{\pi}{2},0), (\frac{3\pi}{2},0)$ (4)

$$\frac{\partial^2 f}{\partial x^2} = (-\sin x)(y^2-1), \quad \frac{\partial^2 f}{\partial y^2} = 2\sin x, \quad \frac{\partial^2 f}{\partial y\,\partial x} = (2y)(\cos x)$$

So

$$D = 2(1-y^2)\sin^2 x - 4y^2\cos^2 x \quad (4)$$

$D(\pi,1) = -4 < 0$ thus Saddle pt. (2)

$D(\pi,-1) = -4 < 0$ thus Saddle pt (2)

$D(\frac{\pi}{2},0) = 2 > 0$ since $\frac{\partial^2 f}{\partial y^2}(\frac{\pi}{2},0) = 2 > 0$ Local MIN (2)

$D(\frac{3\pi}{2},0) = 2 > 0$ since $\frac{\partial^2 f}{\partial y^2}(\frac{3\pi}{2},0) = -2 < 0$ Local Max (2)

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(15) 6. Use the <u>method of Lagrange multipliers</u> to find the largest value and the smallest value for $f(x,y,z) = 2z + xy$ on the sphere $\underbrace{x^2 + y^2 + z^2}_{g(x,y,z)} = 22$

$\nabla f = y\vec{i} + x\vec{j} + 2\vec{k}$, $\nabla g = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

Thus

$\boxed{y = 2x\lambda,\ x = 2y\lambda,\ 2 = 2z\lambda,\ x^2+y^2+z^2 = 22}$ 5

$x=0$ gives $y=0$ and $y=0$ gives $x=0$

which give $(0,0,\pm\sqrt{22})$

Suppose $x \neq 0$ and $y \neq 0$. Then $\frac{y}{x} = 2\lambda = \frac{x}{y}$.

Thus $y^2 = x^2$ so $y = \pm x$.

$y=x$ gives $\lambda = \frac{1}{2}$ so $z=2$ and then $2x^2+4=22$

so $x^2 = 9$ and $x = \pm 3$

$y=-x$ gives $\lambda = -\frac{1}{2}$ so $z=-2$ and $2x^2+4=22$ so $x = \pm 3$

$\boxed{\text{CP's}}$ $(0,0,\pm\sqrt{22})$, $(3,3,2)$, $(-3,-3,2)$, $(3,-3,-2)$, $(-3,3,-2)$ 6

$f(0,0,\sqrt{22}) = 2\sqrt{22} \simeq 9.38$

$f(0,0,-\sqrt{22}) = -2\sqrt{22} \simeq -9.38$

$f(3,3,2) = 4+9 = 13$ LARGEST

$f(-3,-3,2) = 13$ VALUE

$f(3,-3,-2) = -4-9 = -13$

$f(-3,3,-2) = -13$ SMALLEST VALUE

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