CALCULUS 3 EXAM 3

SPRING 2010

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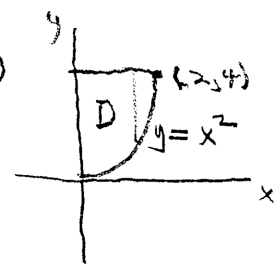
REC. INSTR.________________

REC. TIME________________

TO RECEIVE CREDIT YOU MUST SHOW YOUR WORK

(15) 1. Find the volume of the 3D-region which is under $z = 1 + x$ and above the region in the first quadrant of the xy-plane that is enclosed by $y = x^2$, $y = 4$ and $x = 0$.

$$\text{Volume} = \iint_D (1+x)\,dA \quad \Big\} \ 4$$



$$= \int_0^2 \int_{x^2}^4 (1+x)\,dy\,dx \quad \Big\} \ 5$$

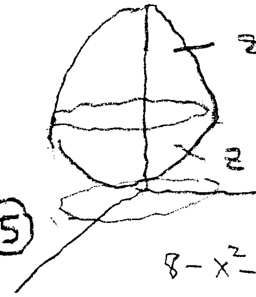
$$= \int_0^2 (y + xy)\Big|_{y=x^2}^{y=4} dx = \int_0^2 [(4+4x) - (x^2 + x^3)]\,dx$$

$$= 4x + 2x^2 - \frac{x^3}{3} - \frac{x^4}{4}\Big|_0^2 \qquad 6$$

$$= 8 + 8 - \frac{8}{3} - 4 = 12 - \frac{8}{3} = \frac{28}{3}$$

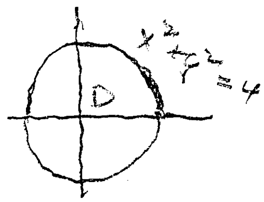
NAME________________

(15) 2. Find the volume of the 3D- region which is enclosed by the surfaces $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$.



$8 - x^2 - y^2 = x^2 + y^2$

gives $x^2 + y^2 = 4$



$$\text{Volume} = \iint_D [(8 - x^2 - y^2) - (x^2 + y^2)]\, dA \quad \Big\}\ 5$$

$$= \int_0^{2\pi}\int_0^2 (8 - 2r^2)\, r\, dr\, d\theta \quad \Big\}\ 5$$

$$= \int_0^{2\pi}\int_0^2 (8r - 2r^3)\, dr\, d\theta$$

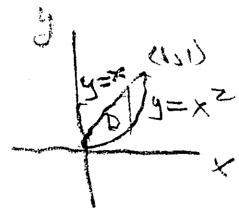
$$= \int_0^{2\pi} \left(4r^2 - \frac{r^4}{2}\right)\Big|_{r=0}^{r=2} d\theta$$

$$= \int_0^{2\pi} (16 - 8)\, d\theta = 16\pi \quad \Big\}\ 5$$

NAME__________

(15) 3. A mass distribution occupies the 3D-region in the first octant which is under $z = 1 + y$ and above the xy-plane and has the surfaces $y = x$ and $y = x^2$ as its sides. The mass density function is $\delta = 24x$ units of mass/unit volume. Calculate the total mass in the region.

$$\text{Mass} = \underbrace{\iiint_T 24x\, dV}_{3} = \underbrace{\iint_D \int_0^{1+y} 24x\, dz\, dA}_{3}$$



$$= \iint_D 24x\, z \Big|_{z=0}^{z=1+y} dA = \iint_D (24x + 24xy)\, dA \quad \Big\}\ 2$$

$$= \int_0^1 \int_{x^2}^{x} (24x + 24xy)\, dy\, dx \quad \Big\}\ 3$$

$$= \int_0^1 24xy + 12xy^2 \Big|_{y=x^2}^{y=x} dx$$

$$= \int_0^1 [(24x^2 + 12x^3) - (24x^3 + 12x^5)]\, dx$$

$$= 8x^3 + 3x^4 - 6x^4 - 2x^6 \Big|_0^1$$

$$= 8 + 3 - 6 - 2 = 3 \text{ units of mass}$$

4

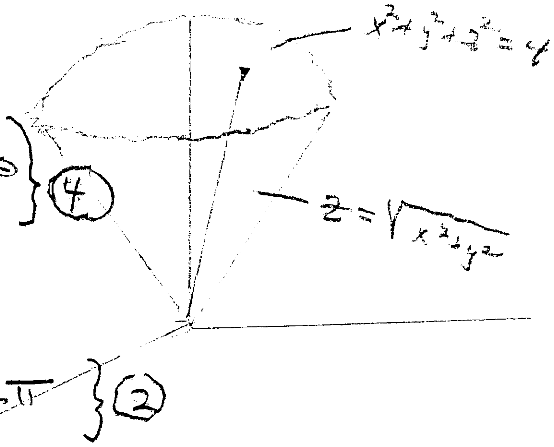
(25) 4. A mass distribution occupies the 3D-region which is inside the sphere $x^2 + y^2 + z^2 = 4$ and above the surface $z = (x^2+y^2)^{\frac{1}{2}}$.
The mass density function is $\delta = z$ units of mass/unit volume.
Find the total mass in the region and then find the z-coordinate of the center of mass. USE SPHERICAL COORDINATES TO EVALUATE THE INTEGRALS.

a) total mass $= \iiint_T z\,dV \;\Big\}\; 4$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{2} (\rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta \;\Big\}\; 4$$

$$= 4\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \cos\phi\,\sin\phi\, d\phi\, d\theta \;\Big\}\; 2$$

$$= 4\int_0^{2\pi} \frac{1}{2}\sin^2\phi \Big|_{\phi=0}^{\phi=\frac{\pi}{4}} d\theta = 4\int_0^{2\pi}\frac{1}{4}\,d\theta = 2\pi \;\Big\}\; 2$$



b) z-coordinate of center of mass $= \frac{1}{2\pi}\iiint_T z\cdot z\,dV \;\Big\}\; 4$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{2} \rho^2\cos^2\phi\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta \;\Big\}\; 4$$

$$= \frac{1}{2\pi}\cdot\frac{32}{5}\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \cos^2\phi\,\sin\phi\, d\phi\, d\theta \;\Big\}\; 1$$

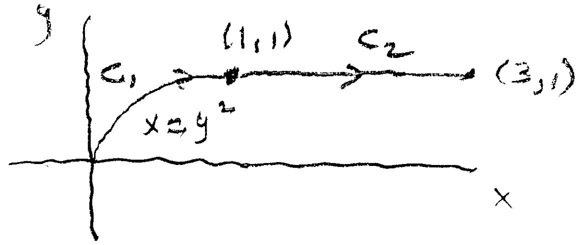
$$= \frac{1}{2\pi}\cdot\frac{32}{5}\int_0^{2\pi} -\frac{1}{3}\cos^3\phi\Big|_{\phi=0}^{\phi=\frac{\pi}{4}} d\theta \;\Big\}\; 2$$

$$= \frac{1}{2\pi}\cdot\frac{32}{15}\int_0^{2\pi}\left[-\frac{1}{2\sqrt{2}}+1\right] d\theta$$

$$= \frac{32}{15}\left(1-\frac{1}{2\sqrt{2}}\right) \simeq 1.379 \;\Big\}\; 2$$

NAME____________

(20) 5. The force field $\mathbf{F} = xy\,\mathbf{i} - x^2\,\mathbf{j}$ acts on an object as it moves in the xy-plane. Calculate the work done by $\mathbf{F}$ as the object moves from (0,0) to (3,1) by first going along $x = y^2$ from (0,0) to (1,1) and then along the horizontal line segment from (1,1) to (3,1).



$$\text{Work} = \int_C xy\,dx - x^2\,dy \quad (4)$$

C_1: $x = t^2$, $y = t$, $0 \le t \le 1$

$dx = 2t\,dt$, $dy = dt$

$$\int_{C_1} = \int_0^1 [(t^3)(2t) - t^4]\,dt = \int_0^1 t^4\,dt = \frac{1}{5} \quad (8)$$

C_2: $x = t$, $y = 1$, $1 \le t \le 3$

$dx = dt$, $dy = 0$

$$\int_{C_2} = \int_1^3 t\,dt = \frac{t^2}{2}\Big|_1^3 = \frac{9}{2} - \frac{1}{2} = 4 \quad (6)$$

Total Work = $4 + \frac{1}{5}$ units of work. (2)

NAME________________

(10) 6. Show that the force field $\mathbf{F} = (2x + 1/y)\,\mathbf{i} + (3y^2 - x/y^2)\,\mathbf{j}$ is conservative in the region $y > 0$ by finding a potential function for it. Then use this potential function to calculate the work done by $\mathbf{F}$ as it acts on an object which moves from (2,1) to (4,2) along some curve in the upper half of the xy-plane.

Potential function

$$f(x,y) = x^2 + x/y + y^3 \quad \} \; 5$$

$$\text{Work} = f(4,2) - f(2,1)$$

$$= (16+2+8) - (4+2+1)$$

$$= 26 - 7 = 19 \text{ units of work} \quad \} \; 5$$