Calculus 3 – EXAM 2

SPRING 2010

88–100 A; 10.8 %

75 – 87 B; 25.8 %

62 – 74 C; 29.4 %

50–61 D; 18.5 %

0 – 49 F; 15.5 %

OF SCORES = 194

Median SCORE = 70

CALCULUS 3 EXAM 2 NAME________________

SPRING 2010 REC. INSTR.________________

TO RECEIVE CREDIT YOU MUST SHOW YOUR WORK

(15) 1. An object is moving in the xy-plane along the curve $y = x^3/3$. It is moving from left to right at a constant speed of 2 ft/sec.

$\frac{dy}{dx} = x^2$

$\frac{d^2y}{dx^2} = 2x$

a) find a_T and a_N when the object is at $(x, x^3/3)$

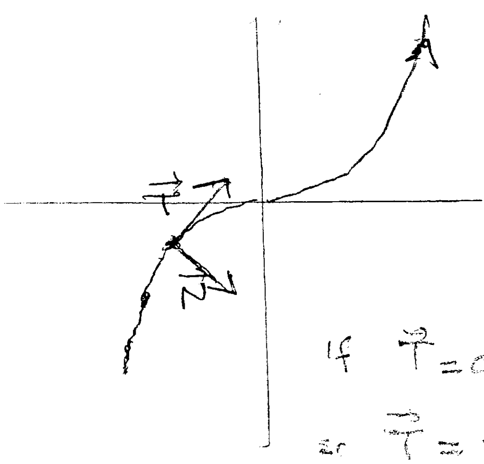
$a_T = \frac{d}{dt}(\text{SPEED}) = 0$ since SPEED = 2 is CONSTANT (3)

$a_N = K(\text{SPEED})^2 = 4K$

$$K = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} = \frac{|2x|}{[1+x^4]^{3/2}}$$

so $\boxed{a_N = \frac{8|x|}{[1+x^4]^{3/2}}}$ (4)

b) find the velocity vector and the acceleration vector when the object is at $(-1, -1/3)$



$\vec{v} = (\text{SPEED})\vec{T} = 2\vec{T}$

when $x = -1$

$$\text{slope } \vec{T} = \left.\frac{dy}{dx}\right|_{x=-1} = (-1)^2 = 1$$

if $\vec{T} = a\vec{i} + b\vec{j}$, $a^2+b^2 = 1$ and $\frac{b}{a} = 1$

so $\vec{T} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$

and $\boxed{\vec{v} = 2\vec{T} = \sqrt{2}\,\vec{i} + \sqrt{2}\,\vec{j}}$ (4)

$$\vec{a} = \underbrace{a_T}_{0}\vec{T} + a_N\vec{N} = \left.\frac{8|x|}{[1+x^4]^{3/2}}\right|_{x=-1}\vec{N} = \frac{8}{2\sqrt{2}}\vec{N} = \frac{4}{\sqrt{2}}\vec{N}$$

$\vec{N} = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$ (from $\vec{T}$)

so $\boxed{\vec{a} = \frac{4}{\sqrt{2}}\vec{N} = 2\vec{i} - 2\vec{j}}$ (4)

NAME____________

(10) 2. Find the equation of the tangent plane to the surface

$$x^2y^3 - 4xy^2 + yz^2 + xz = 0$$

at the point (1,1,1) .

$F(x,y,z) = x^2y^3 - 4xy^2 + yz^2 + xz$

$F_x = 2xy^3 - 4y^2 + z = 2 - 4 + 1 = -1$ at $(1,1,1)$

$F_y = 3x^2y^2 - 8xy + z^2 = 3 - 8 + 1 = -4$ at $(1,1,1)$

$F_z = 2yz + x = 2 + 1 = 3$ at $(1,1,1)$

5

Thus $\vec{n} = \langle -1, -4, 3 \rangle$

and equation of tangent plane at (1,1,1) is

$-1(x-1) - 4(y-1) + 3(z-1) = 0$

$-x - 4y + 3z + 2 = 0$

5

NAME__________

(15) 3. Suppose $z = f(x,y)$ and $x = r\cos\theta$, $y = r\sin\theta$.
Thus z can be considered to be a function of r and θ .
Also suppose that when $x = 1$ and $y = 2$,

$$f_x(1,2) = 4 \quad \text{and} \quad f_y(1,2) = 3 .$$

Use the chain rule to find the partial derivative of z with respect to θ and the partial derivative of z with respect to r when $x = 1$ and $y = 2$.

$$\frac{\partial z}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = \boxed{\frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta)} \quad 4$$

At $(x,y) = (1,2)$

$$\frac{\partial z}{\partial \theta} = 4(\underbrace{-r\sin\theta}_{y}) + 3(\underbrace{r\cos\theta}_{x}) = -4y + 3x = -8+3 = -5 \Big\} \quad 4$$

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \boxed{\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta} \quad 4$$

At $(x,y) = (1,2)$, $r = \sqrt{x^2+y^2} = \sqrt{5}$

$$\left.\begin{aligned}\frac{\partial z}{\partial r} &= 4\cos\theta + 3\sin\theta = 4\left(\frac{x}{r}\right) + 3\left(\frac{y}{r}\right) \\ &= 4\left(\frac{1}{\sqrt{5}}\right) + 3\left(\frac{2}{\sqrt{5}}\right) = \frac{10}{\sqrt{5}} = 2\sqrt{5}\end{aligned}\right\} \quad 3$$

NAME____________

(20) 4. Let $f(x,y) = 5y(x^2 + 4y)^{\frac{1}{2}}$.

a) find the gradient vector field of f(x,y)

$$\nabla f = (5y)(\tfrac{1}{2})(x^2+4y)^{-\frac{1}{2}}\,2x\,\vec{i} + \left[5(x^2+4y)^{\frac{1}{2}} + (5y)(\tfrac{1}{2})(x^2+4y)^{-\frac{1}{2}}\,4\right]\vec{j}$$

$$\nabla f = \frac{5xy\ \ 2}{\sqrt{x^2+4y}}\,\vec{i} + \left[5\sqrt{x^2+4y} + \frac{10y}{\sqrt{x^2+4y}}\right]\vec{j}$$ 5

b) find the directional derivative of f(x,y) at (x,y) = (3,4) in the direction of the vector **v** = 2**i** - **j**

$|\vec{v}| = \sqrt{5}$ so $\vec{u} = \frac{2}{\sqrt{5}}\vec{i} - \frac{1}{\sqrt{5}}\vec{j}$ 2

$\nabla f(3,4) = 12\vec{i} + [25 + 8]\vec{j} = 12\vec{i} + 33\vec{j}$ 3

$D_u f(3,4) = \nabla f(3,4)\cdot\vec{u} = \frac{24}{\sqrt{5}} - \frac{33}{\sqrt{5}} = -\frac{9}{\sqrt{5}}$ 5

c) find the value of the largest directional derivative of f(x,y) at (3,4)

Largest Dir. Deriv at (3,4)

$= |\nabla f(3,4)| = \sqrt{(12)^2 + (33)^2} = 35.114$ 5

NAME___________

(25) 5. Let $f(x,y) = xy^2 - 2x^2y + 2x^2 - x$. Find all the critical points of f and then apply the 2nd partials test to each critical point to determine its nature.

$f_x = y^2 - 4xy + 4x - 1 = 0, \quad f_y = 2xy - 2x^2 = 0$ (6)

$(2x)(y-x) = 0$

so $x = 0$ OR $y = x$

$x = 0$ gives $y^2 - 1 = 0$

so $y = \pm 1$.

$y = x$ gives $x^2 - 4x^2 + 4x - 1 = 0$

$-3x^2 + 4x - 1 = 0$

$3x^2 - 4x + 1 = 0$

$x = \dfrac{4 \pm \sqrt{16-12}}{6} = \dfrac{4 \pm 2}{6} = 1, \frac{1}{3}$

Thus CP's ARE $(x,y) = (0,1), (0,-1), (1,1), (\frac{1}{3}, \frac{1}{3})$ (6)

$f_{xx} = -4y + 4, \quad f_{yy} = 2x, \quad f_{xy} = 2y - 4x$

So $\boxed{D = (2x)(4-4y) - (2y-4x)^2}$ (6)

$D(0,1) = -4 < 0$ thus Saddle pt (2)

$D(0,-1) = -4 < 0$ thus Saddle pt (2)

$D(1,1) = -4 < 0$ thus Saddle pt (2)

$D(\frac{1}{3}, \frac{1}{3}) = (\frac{2}{3})(4 - \frac{4}{3}) - (\frac{2}{3} - \frac{4}{3})^2 = (\frac{2}{3})(\frac{8}{3}) - \frac{4}{9} = \frac{16}{9} - \frac{4}{9} > 0$

Since also $f_{yy}(\frac{1}{3}, \frac{1}{3}) = \frac{2}{3} > 0$ Local MIN. (3)

NAME________________

(15) 6. Use the method of Lagrange multipliers to find the largest value and the smallest value for $f(x,y,z) = 4z + 2xy$ on the sphere $\underbrace{x^2 + y^2 + z^2}_{g(x,y,z)} = 16$.

$\nabla f = 2y\vec{i} + 2x\vec{j} + 4\vec{k}$, $\lambda \nabla g = 2\lambda x\vec{i} + 2\lambda y\vec{j} + 2\lambda z\vec{k}$

so $\boxed{2y = 2\lambda x,\ 2x = 2\lambda y,\ 4 = 2\lambda z,\ x^2+y^2+z^2 = 16}$ 6

$x=0$ gives $y=0$ gives $z^2=16$, $z=\pm 4$

$y=0$ gives $x=0$ gives $z=\pm 4$

$x = \lambda y = \lambda^2 x$

$x(1-\lambda^2) = 0$

Suppose $x \neq 0$ AND $y \neq 0$

Then $\frac{y}{x} = \lambda = \frac{x}{y}$ so $y^2 = x^2$ and $y = \pm x$

$y = x$ gives $\lambda = 1$ so $z = 2$ and $2x^2 + 4 = 16$

$2x^2 = 12$, $x^2 = 6$, $x = \pm\sqrt{6}$

$y = -x$ gives $\lambda = -1$ so $z = -2$ and $x = \pm\sqrt{6}$

Thus CP's are $(x,y,z) = (0,0,4), (0,0,-4), (\sqrt{6},\sqrt{6},2), (-\sqrt{6},-\sqrt{6},2), (\sqrt{6},-\sqrt{6},-2), (-\sqrt{6},\sqrt{6},-2)$ 6

$f(0,0,4) = 16$

$f(0,0,-4) = -16$

$f(\sqrt{6},\sqrt{6},2) = f(-\sqrt{6},-\sqrt{6},2) = 8+12 = 20$ ← LARGEST VALUE

$f(\sqrt{6},-\sqrt{6},-2) = f(-\sqrt{6},\sqrt{6},-2) = -8-12 = -20$ ← SMALLEST VALUE

3