Mechanical Vibrations

Ingredients.

1. From physics we have *Newton's Law of motion*: Force equals mass times acceleration.

$$m\frac{d^2x}{dt^2} \;=\; \text{Force}$$

2. From physics (or engineering) we have *Hooke's law of the spring*: Spring force is proportional to the negative of the extension of the spring from the equilibrium position.

$$\text{Spring force} \;=\; -kx$$

3. From physics (or engineering) we have a mathematical model of *friction forces*: friction is proportional to the negative of the velocity.

$$\text{Friction force} \;=\; -c\frac{dx}{dt}$$

4. We may add an *external driving force*, $F(t)$.

Recipes.

1. **Free undamped motion.** Combine 1 and 2. This motion is described by the ODE

$$mx'' \;=\; -kx$$

which rewrites as a homogeneous, linear second order ODE in standard form

$$x'' + \omega_0^2 x \;=\; 0$$

where $\omega_0 = \sqrt{k/m}$ is called the *circular frequency* of the system.
The characteristic equation

$$r^2 + \omega_0^2 \;=\; 0$$

has solutions $r = \pm\omega_0 i$. Therefore the ODE has solution

$$x(t) \;=\; c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$$

which rewrites using trig addition formulae as

$$x(t) \;=\; A\cos(\omega_0 t - \varphi)$$

Note that

- This is a sinusoid.
- The motion is called *simple harmonic motion.*
- The coefficient A is called the *amplitude* of the motion.
- The motion is periodic with *period* $T = 2\pi/\omega_0$ seconds.
- The *frequency* of this motion is $1/T = \omega_0/2\pi$ Hz.
- The angle φ is called the *phase shift* or *phase angle.*

2. **Free damped motion.** Combine 1, 2 and 3. This motion is described by the ODE

$$mx'' = -kx - cx'$$

which rewrites as a homogeneous, linear second order ODE in standard form

$$x'' + 2px' + \omega_0^2 x = 0$$

where $\omega_0 = \sqrt{k/m}$ as before and $p = k/2m$.
The characteristic equation

$$r^2 + 2pr + \omega_0^2 = 0$$

has solutions given by the quadratic formula

$$r = -p \pm \sqrt{p^2 - \omega_0^2}$$

There are thee cases to explore, depending on the sign of $p^2 - \omega_0^2$.

(a) Case $p^2 - \omega_0^2 > 0$. This is called *over damped motion.* The solution is

$$x(t) = c_1 e^{-(p+\sqrt{p^2-\omega_0^2})t} + c_2 e^{-(p-\sqrt{p^2-\omega_0^2})t}$$

These functions are graphed in Figure 3.4.7 of your text. Notice that $x(t) \to 0$ without oscillating as $t \to \infty$.

(b) Case $p^2 - \omega_0^2 = 0$. This is called *critically damped motion.* The root $r = -p$ has multiplicity 2, and so the solution is

$$x(t) = c_1 e^{-pt} + c_2 t e^{-pt}$$

These functions are graphed in Figure 3.4.8 of your text. Notice that $x(t) \to 0$ without oscillating as $t \to \infty$.

(c) Case $p^2 - \omega_0^2 < 0$. This is called *underdamped motion.* The roots are complex conjugate pairs $r = -p \pm \omega_1 i$ where $\omega_1 = \sqrt{\omega_0^2 - p^2}$, and so the solution is

$$x(t) = c_1 e^{-pt}\cos(\omega_1 t) + c_2 e^{-pt}\sin(\omega_1 t)$$

As in the undamped case, this can be rewritten using trig identities to get

$$x(t) = Ae^{-pt}\cos(\omega_1 t - \varphi)$$

The graph is shown in Figure 3.4.9 of your text. Note that $x(t) \to 0$ as $t \to \infty$ but that the function oscillates with frequency $\omega_1/2\pi$. The motion is not actually periodic since the amplitude Ae^{-pt} is decreasing in time, but you can speak of a pseudoperiod $T = 2\pi/\omega_1$.

3. **Forced undamped motion.** Combine 1, 2 and 4. Suppose that the external driving force varies periodically in time, described by the function

$$F(t) = F_0\cos(\omega t).$$

The ODE describing the motion is

$$mx'' = -kx + F(t)$$

which rewrites as a linear second order ODE in standard form

$$x'' + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$$

There are now two cases to consider.

(a) Case $\omega \neq \omega_0$. In this case the frequency of the periodic external driving force is **different** from the natural frequency of the system. We use the method of undetermined coefficients to find a particular solution which is a linear combination of $\cos(\omega t)$ and $\sin(\omega t)$. This turns out to be

$$x_p(t) = \frac{F_0/m}{\omega_0^2 - \omega^2}\cos(\omega t)$$

The general solution is

$$x(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0/m}{\omega_0^2 - \omega^2}\cos(\omega t)$$

A typical graph is shown in Figure 3.6.2 of your text.
In the case where $x(0) = 0 = x'(0)$ we obtain the following solution

$$x(t) = \frac{F_0/m}{\omega_0^2 - \omega^2}(\cos(\omega t) - \cos(\omega_0 t))$$

Some more trig identities can be used to show that this is the same as

$$x(t) = \frac{2F_0/m}{\omega^2 - \omega_0^2}\sin((\omega - \omega_0)t/2)\sin((\omega + \omega_0)t/2)$$

In the special case where driving force frequency ω is almost equal to the natural frequency ω_0, then $(\omega - \omega_0)/2$ is tiny in comparison with $(\omega + \omega_0)/2$. We can think of the solution as a signal with pseudofrequency $(\omega + \omega_0)/2$ and slowly varying amplitude given by

$$\frac{2F_0/m}{\omega^2 - \omega_0^2}\sin((\omega - \omega_0)t/2)$$

A graph is shown in Figure 3.6.3 of the text. This is the phenomenon of *beats*.

(b) Case $\omega = \omega_0$. In this case the frequency of the periodic external driving force is the **same** as the natural frequency of the system. We use the method of undetermined coefficients to find a particular solution. This will be a linear combination of $\cos(\omega_0 t)$, $\sin(\omega_0 t)$, $t\cos(\omega_0 t)$, $t\sin(\omega_0 t)$, but we remember to remove the combinations of $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$. We get

$$x_p(t) = \frac{F_0}{2m\omega_0}t\sin(\omega_0 t).$$

The general solution of the equation is

$$x(t) = \frac{F_0}{2m\omega_0}t\sin(\omega_0 t) + c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$$

For example, the solution which satisfies the initial conditions $x(0) = 0 = x'(0)$ is

$$x(t) = \frac{F_0}{2m\omega_0}t\sin(\omega_0 t).$$

A graph is shown in Figure 3.6.4 of your text. Notice that the amplitude of the motion $\frac{F_0}{2m\omega_0}t$ tends to ∞ as $t \to \infty$. This phenomenon is called *pure resonance*. In reality the system (spring etc) would distort and break before the amplitude gets too large.

Remark. Note that the equation describing beats limits on the pure resonance solution as $\omega \to \omega_0$. Indeed the amplitude portion of the beats solution is

$$\frac{2F_0/m}{\omega^2 - \omega_0^2}\sin((\omega - \omega_0)t/2) = \frac{2F_0 t}{2m(\omega + \omega_0)}\frac{\sin((\omega - \omega_0)t/2)}{(\omega - \omega_0)t/2}$$

By the famous limit $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$, we see that this amplitude function tends to the function

$$\frac{F_0 t}{2m\omega_0}$$

as $\omega \to \omega_0$ provided t is not very large. This is the amplitude expression in the pure resonance function. Again, in practice the spring will deform or break before the amplitude gets too high, and the phenomenon is generally termed resonance whether it occurs because $\omega=\omega_0$ or because ω is close enough to ω_0 to cause system failure.

4. **Forced damped motion.** Combine 1, 2, 3 and 4. Suppose that the external driving force varies periodically in time, described by the function

$$F(t) \;=\; F_0\cos(\omega t).$$

The ODE describing the motion is

$$mx'' \;=\; -kx - cx' + F(t)$$

which rewrites as a linear second order ODE in standard form

$$x'' + 2px' + \omega_0^2 x \;=\; \frac{F_0}{m}\cos(\omega t) \qquad\qquad (**)$$

The solution of this equation is of the form

$$x(t) \;=\; x_h + x_p$$

where x_h is the solution to the homogeneous equation (already obtained in the free damped motion case above — item 2). We had 3 possible functions for x_h depending on whether $p^2-\omega_0^2<0$, $p^2-\omega_0^2=0$, or $p^2-\omega_0^2>0$. In all three cases, the solution had the property that $x_h(t)\to 0$ as $t\to\infty$. This portion of the solution is called **transient**.

The remaining portion of the solution x_p will be found by the method of undetermined coefficients, and will be closely related to the forcing function. This will not die off as $t\to\infty$, and is called the **steady-state** portion of the solution. We look for coefficients A and B so that

$$x_p(t) \;=\; A\cos(\omega t) + B\sin(\omega t)$$

is a solution to the ODE $(**)$. Substituting for y_p, y_p' and y_p'' and, after tears of joy/frustration, we obtain

$$x_p(t) \;=\; \frac{(k-m\omega^2)F_0}{(k-m\omega^2)^2+(c\omega)^2}\cos(\omega t) + \frac{c\omega F_0}{(k-m\omega^2)^2+(c\omega)^2}\sin(\omega t)$$

This can be rewritten using trig identities as

$$x_p(t) \;=\; C\cos(\omega t - \varphi)$$

where the amplitude is given by the expression

$$C \;=\; \frac{F_0}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}.$$

and the phase angle satisfies $\tan(\varphi)=(c\omega)/(k-m\omega^2)$.

Note that the amplitude is always finite, but for a fixed system (i.e., fixed k, m and c) we can vary the frequency of the external driving force to obtain a maximum amplitude. Mathematically we just find the value of ω that optimizes C; solve $\frac{dC}{d\omega}=0$ to get a maximum amplitude when the driving force frequency is

$$\omega \;=\; \sqrt{\frac{k}{m}-\frac{c^2}{2m^2}}$$

Summary — Mechanical Vibrations

Motion	**ODE**	**Solution**	**Comments**
Free undamped	$mx'' + kx = 0$	$x(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$ $= C \cos(\omega_0 t - \varphi)$	$\omega_0^2 = k/m$ is the system frequency
Free damped	$x'' + 2px' + \omega_0^2 x = 0$ $p = c/2m$	$x(t) = c_1 e^{-(p+\sqrt{p^2-\omega_0^2})t} + c_2 e^{-(p-\sqrt{p^2-\omega_0^2})t}$	overdamped $p^2 - \omega_0^2 > 0$
		$x(t) = c_1 e^{-pt} + c_2 t e^{-pt}$	critically damped $p^2 - \omega_0^2 = 0$
	$\omega_1 = \sqrt{\omega_0^2 - p^2}$	$x(t) = e^{-pt}(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t))$	underdamped $p^2 - \omega_0^2 < 0$
Forced undamped	$mx'' + kx = F_0 \cos(\omega t)$ $x'' + \omega_0^2 x = \frac{F_0}{m} \cos(\omega t)$	$x(t) = x_h + x_p$ $x_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$ $x_h = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$ $x(t) = \frac{2F_0/m}{\omega^2 - \omega_0^2} \sin(\frac{(\omega-\omega_0)t}{2}) \sin(\frac{(\omega+\omega_0)t}{2})$	Periodic external driving force $\omega \neq \omega_0$ satisfies $x'(0) = 0 = x(0)$ beats and resonance
		$x(t) = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t)$	$\omega = \omega_0$ $x'(0) = 0 = x(0)$ pure resonance
Forced damped	$mx'' + cx' + kx = F_0 \cos(\omega t)$	$x(t) = x_h + x_p$ $x_h =$ free damped sol. $x_p = C \cos(\omega t - \varphi)$ $C = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}$ $\tan\varphi = (c\omega)/(k - m\omega^2)$	Periodic external driving force transient steady-state