First order ODEs — Substitution Techniques

We have been considering ODEs of the form

$$\frac{dy}{dx} \;=\; F(x,y).$$

We know how to handle

- *separable equations* (ones where $F(x,y) = f(x)g(y)$),
- *first order linear equations* (ones where $F(x,y) = q(x) - p(x)y$), and
- *exact equations* (ones where $F(x,y) = -\frac{M(x,y)}{N(x,y)}$ and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$).

Now we can add several more families of equations which have the following format

$$\frac{dy}{dx} = f(v)$$

where $v = v(x,y)$ is a particularly nice function of x and y. In all cases you can use the chain rule (or implicit differentiation) to write $\frac{dv}{dx}$ in terms of x, y and y'.

$$\frac{dv}{dx} \;=\; \frac{\partial v}{\partial x}\frac{dx}{dx} + \frac{\partial v}{\partial y}\frac{dy}{dx} \;=\; \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx}$$

The result will be an ODE in v and x. This should remind you somewhat of *substitution techniques* for computing antiderivatives in calculus. Here are some nice functions $v(x,y)$.

Example 1. $v(x,y) = ax + by + c$**.** Here a, b, c are constants.
In this case

$$\frac{dv}{dx} \;=\; \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} \;=\; a + b\frac{dy}{dx}$$

and the ODE becomes

$$\frac{dv}{dx} \;=\; a + b\frac{dy}{dx} \;=\; a + bf(v)$$

or, simply

$$\frac{dv}{dx} \;=\; a + bf(v).$$

This is separable

$$\frac{dv}{a + bf(v)} \;=\; dx.$$

Example 2. $v(x,y) = \frac{y}{x}$**.** These are called *homogenous equations.*
In this case

$$\frac{dv}{dx} \;=\; -\frac{y}{x^2} + \frac{1}{x}\frac{dy}{dx} \;=\; -\frac{v}{x} + \frac{1}{x}\frac{dy}{dx}$$

and multiplying across by x gives

$$\frac{dy}{dx} \;=\; v + x\frac{dv}{dx}$$

Therefore the ODE $y' = f(y/x)$ becomes

$$v + x\frac{dv}{dx} \;=\; f(v)$$

which is separable

$$\frac{dv}{f(v) - v} \;=\; \frac{dx}{x}.$$

Sometimes substitutions are found after manipulating the equation a little bit, and the substitution may not fit into the general rubric $y' = f(v)$ above. However the way of computing $\frac{dv}{dx}$ is no different from before, and the general strategy of reducing to a simpler first order ODE in v is the same.

Example 3. Bernoulli Equations. The following equations (called Bernoulli equations) are a classic example.

$$\frac{dy}{dx} + p(x)y \;=\; q(x)y^n$$

Note that if $n = 0$ or $n = 1$ then the Bernoulli equation becomes a first order linear equation. What about other (real) values of n? First divide across by y^n to get

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{p(x)}{y^{n-1}} \;=\; q(x)$$

The leftmost term of this equation is almost the output of a chain rule

$$\frac{d}{dx}\left(\frac{1}{y^{n-1}}\right) \;=\; (1-n)\frac{1}{y^n}\frac{dy}{dx}$$

and the second term contains $\frac{1}{v^{(n-1)}}$, so the substitution $v = \frac{1}{y^{n-1}}$ makes sense. The ODE becomes

$$\frac{1}{(1-n)}\frac{dv}{dx} + p(x)v \;=\; q(x)$$

or simply

$$\frac{dv}{dx} + (1-n)p(x)v \;=\; (1-n)q(x).$$

This is a first order linear ODE in v. We know how to solve these.

Second Order Equations. Finally, there are a few second order ODEs which can be solved by making a substitution which reduces them to a first order ODE. Solve the first order equation, and do one final integration. Here are two general instances where this strategy works.

Case: y does not appear explicitly in the 2nd order ODE. Consider a second order ODE of the form

$$F(x, y', y'') = 0$$

Substitute $v = y'$. Then we get a first order ODE of the form

$$F(x, v, v') = 0$$

If we can solve this for $v = v(x)$ then simply integrate to get $y = \int v(x)\, dx + C$.

Case: x does not appear explicitly in the 2nd order ODE. Consider a second order ODE of the form

$$F(y, y', y'') = 0$$

where derivatives are with respect to x. Substitute $v = y'$. Then

$$y'' \;=\; \frac{dv}{dx} \;=\; \frac{dv}{dy}\frac{dy}{dx} \;=\; v\frac{dv}{dy}$$

and we get a first order ODE of the form

$$F(y, v, vv') \;=\; 0.$$

Given a solution $v = v(y)$ we can write

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{v}$$

and so

$$x = \int \frac{dy}{v} + C$$

gives the desired functional relationship between x and y.

Remark. Famous examples of the last type of second order equation occur when considering motion under a force which does not depend on time. Note that here the variable x will represent time, and so t is used in place of x. One example is

$$m\frac{d^2y}{dt^2} = -mg$$

motion under gravity.

Another example is

$$m\frac{d^2y}{dt^2} = -ky$$

motion under a spring (Hooke's law).

Substitute $v = \frac{dy}{dt}$ (note that physically v is velocity) and rewrite

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}.$$

The gravity motion equation becomes

$$v\frac{dv}{dy} = -g$$

which integrates (w.r.t y) to give

$$\frac{v^2}{2} + gy = C$$

Multiplying across by m gives the law of conservation of energy; kinetic energy plus gravitational potential energy remains constant.

The spring motion equation becomes

$$v\frac{dv}{dy} = -\frac{k}{m}y$$

which integrates (w.r.t y) to give

$$\frac{v^2}{2} + \frac{ky^2}{2m} = C$$

Multiplying across by m gives the law of conservation of energy; kinetic energy plus spring potential energy remains constant.

These are two instances of the general principle

$$\int \text{(Newton's Law of motion)}\, d\text{(position)} = \text{Law of Conservation of Energy.}$$