Pge 333 : Q2

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{(k_1+k_2)}{M_1} & \frac{k_2}{M_1} \\ \frac{k_2}{M_2} & -\frac{(k_2+k_3)}{M_2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Hwk 16

$$\frac{d\vec{x}}{dt} = A\vec{x} \quad \text{where} \quad A = \begin{pmatrix} -5 & 4 \\ 4 & -5 \end{pmatrix}$$

Find evalues

$\det(A-\lambda I) = 0$ $\qquad (-5-\lambda)(-5-\lambda) - (4)(4) = 0$ $\qquad \lambda^2 + 10\lambda + 25 - 16 = 0$

$(\lambda+1)(\lambda+9) = 0$

Find evectors

$\boxed{\lambda = -1}$

$\begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\qquad -4x + 4y = 0$, $y = x$ $\qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\boxed{\lambda = -9}$

$\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\qquad x + y = 0$ $\qquad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Solutions

$\vec{X}_1(t)$: $\quad e^{\sqrt{-1}\,t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (\cos t + i\sin t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\quad \vec{X}_1(t) = (a_1\cos t + b_1 \sin t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{X}_2(t)$: $\quad e^{\sqrt{-9}\,t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = (\cos 3t + i\sin 3t)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$; $\quad \vec{X}_2(t) = (a_2\cos 3t + b_2 \sin 3t)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Natural frequencies (circular) $\qquad \omega_1 = 1 \qquad \omega_2 = 3$

$\boxed{\text{For } \vec{X}_1(t)}$

$x_1(t) = c_1\cos(t - \alpha_1)$ $\qquad c_1 = \sqrt{a_1^2 + b_1^2}$; $\cos\alpha_1 = \frac{a_1}{c_1}$

$x_2(t) = c_1\cos(t - \alpha_1)$

$\boxed{\text{For } \vec{X}_2(t)}$

$x_1(t) = c_2\cos(3t - \alpha_2)$ $\qquad c_2 = \sqrt{a_2^2 + b_2^2}$;

$x_2(t) = -c_2\cos(3t - \alpha_2)$ $\qquad \cos(\alpha_2) = \frac{a_2}{c_2}$

Page 360 Q22

$$A = \begin{bmatrix} 6 & 4 \\ -9 & -6 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 6 & 4 \\ -9 & -6 \end{bmatrix}\begin{bmatrix} 6 & 4 \\ -9 & -6 \end{bmatrix} = \begin{bmatrix} 36-36 & 24-24 \\ -54+54 & -36+36 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

($A^2 = 0 \Rightarrow A$ is nilpotent)

$$e^{At} = I + At + \frac{A^2t^2}{2} + \cdots$$

$$= I + At + 0 + 0 + \cdots$$

$$= I + At = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{bmatrix} 6t & 4t \\ -9t & -6t \end{bmatrix}$$

$$= \begin{bmatrix} 1+6t & 4t \\ -9t & 1-6t \end{bmatrix}$$

Q25

$$A = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix}^2 = 0$$

$$e^{At} = e^{2It + \begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix}t} = e^{2It} \cdot e^{\begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix}t}$$

$$= \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix} \cdot \left[I + \begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix}t + \begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix}^2 \frac{t^2}{2!} + \cdots \right]$$

$$= e^{2t} I \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 5t \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \cdots \right]$$

$$= e^{2t} \begin{pmatrix} 1 & 5t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{2t} & 5te^{2t} \\ 0 & e^{2t} \end{pmatrix}.$$

e^{At}

solution to IVP

Therefore, $\vec{X}(t) = e^{At}\vec{X}(0) = \begin{pmatrix} e^{2t} & 5te^{2t} \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 4e^{2t} + 35te^{2t} \\ 7e^{2t} \end{pmatrix}$

P 370 : Q 27 $\boxed{\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}(t)}$ — (*) $\vec{f}(t) = \begin{pmatrix} 36t^2 \\ 6t \end{pmatrix}$

$\vec{X}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $A = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}$; $e^{At} = \begin{pmatrix} 1+2t & -4t \\ t & 1-2t \end{pmatrix}$

Choose sol^n $\vec{X}_p$ of the form

$$\vec{X}_p = e^{At}\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} 1+2t & -4t \\ t & 1-2t \end{pmatrix}\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}$$

$$\frac{d\vec{x}_p}{dt} = \frac{d}{dt}\left(e^{At}\vec{C}(t)\right) = Ae^{At}\vec{C}(t) + e^{At}\vec{C}'(t)$$

$$= A\vec{X}_p + e^{At}\vec{c}'(t)$$

Comparing with (*) gives $e^{At}\vec{c}'(t) = \begin{pmatrix} 36t^2 \\ 6t \end{pmatrix}$

$$\begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \begin{pmatrix} 1+2t & -4t \\ t & 1-2t \end{pmatrix}^{-1}\begin{pmatrix} 36t^2 \\ 6t \end{pmatrix}$$

$$= \cancel{\frac{1}{1-4t^2+4t^2}}^{\,1}\begin{pmatrix} 1-2t & 4t \\ -t & 1+2t \end{pmatrix}\begin{pmatrix} 36t^2 \\ 6t \end{pmatrix}$$

$$= \begin{pmatrix} (1-2t)36t^2 + 24t^2 \\ -36t^3 + (1+2t)6t \end{pmatrix} = \begin{pmatrix} 60t^2 - 72t^3 \\ -36t^3 + 12t^2 + 6t \end{pmatrix}$$

$$c_1(t) = \int 60t^2 - 72t^3\,dt = 20t^3 - 18t^4$$

$$c_2(t) = \int -36t^3 + 12t^2 + 6t\,dt = -9t^4 + 4t^3 + 3t^2$$

$$\Rightarrow \vec{X}_p(t) = e^{At}\begin{pmatrix} 20t^3 - 18t^4 \\ -9t^4 + 4t^3 + 3t^2 \end{pmatrix}$$

$\vec{X}_p(0) = I\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $\vec{X}_p(t)$ satisfies the I.V.P.

Solution

$$\vec{X}_p(t) = \begin{pmatrix} 1+2t & -4t \\ t & 1-2t \end{pmatrix} \begin{pmatrix} 20t^3 - 18t^4 \\ -9t^4 + 4t^3 + 3t^2 \end{pmatrix}$$

$$= \begin{pmatrix} (1+2t)(20t^3-18t^4) - 4t(-9t^4+4t^3+3t^2) \\ t(20t^3-18t^4) + (1-2t)(-9t^4+4t^3+3t^2) \end{pmatrix} \leftarrow \text{two polys in } t.$$