Miscellaneous expressions and definitions.

1. **Trig Addition, Half Angle.**

   $\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$ $\cos(2A) = \cos^2(A) - \sin^2(A)$
   $\cos(2A) = 2\cos^2(A) - 1$ $\cos(2A) = 1 - 2\sin^2(A)$ $\sin^2(x) = (1 - \cos(2x))/2$
   $\cos^2(x) = (1 + \cos(2x))/2$ $\sin(2x) = 2\sin(x)\cos(x)$
   $\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$

2. **Hyperbolic.**

   $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$

3. **Integration by Parts.**

   $\int u\,dv \;=\; uv \;-\; \int v\,du$

4. **Integration by substitution.**

   $$\int f(u(x))\frac{du}{dx}\,dx \;=\; \int f(u)\,du$$

5. **Inverse Trig.**

   $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$ $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(\frac{x}{a})$

6. **Trig Substitutions.**

   For $\sqrt{a^2 - x^2}$ use $x = a\sin(\theta)$
   For $\sqrt{a^2 + x^2}$ use $x = a\tan(\theta)$
   For $\sqrt{x^2 - a^2}$ use $x = a\sec(\theta)$

7. **Some integrals.**

   $$\int \frac{dx}{x} \;=\; \ln|x| + C \qquad\qquad \int \tan(x)\,dx \;=\; \ln|\sec(x)| + C$$

   $$\int \sec(x)\,dx \;=\; \ln|\sec(x) + \tan(x)| + C$$

8. **First order linear ODE** $y' + p(x)y = q(x)$ can be solved by first multiplying across by an integrating factor

   $$I \;=\; e^{\int p\,dx}$$

9. The equation $M(x,y)dx + N(x,y)dy = 0$ is said to be **exact** if $M_y = N_x$. If it is exact, it can be solved by antidifferentiating M with respect to x and N with respect to y to obtain $F(x,y)$ and then setting $F(x,y) = C$.

10. An ODE of the form $y' = f(ax + by + c)$ can be solved by first making a substitution $v = ax + by + c$.

11. An ODE of the form $y' = f(y/x)$ can be solved by first making a substitution $v = y/x$.

12. The **Bernoulli equation** $y' + p(x)y = q(x)y^n$ can be solved by first making a substitution $v = \frac{1}{y^{n-1}}$.

13. Some **second order ODEs** can be solved by making the substitution $v = y'$.

14. A linear ODE is of the form
$$Ly \;=\; f(x) \qquad\qquad (*)$$
where L is a linear differential operator
$$L \;=\; y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y.$$
The solution to $(*)$ can be written as
$$y \;=\; y_h + y_p$$
where y_p is a particular solution to $(*)$ and y_h is the solution to the associated homogeneous equation
$$Ly \;=\; 0 \qquad\qquad (+)$$

15. The general solution of the homogeneous equation is of the form
$$y_h \;=\; c_1 y_1 + \cdots + c_n y_n$$
where $y_1, \ldots, y_n$ are linearly independent solutions of $(+)$.

16. In the case the a_i are constant functions the solution of $(+)$ is a sum of exponential terms e^{rx} and polynomial times exponential $x^m e^{rx}$ where the r are solutions of the characteristic equation
$$r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n \;=\; 0.$$
The polynomial times exponential terms occur when the root r repeats with multiplicity greater than 1. Euler's identity
$$e^{(a+ib)x} \;=\; e^{ax}(\cos(bx) + i\sin(bx))$$
is useful for dealing with the case when the roots are complex numbers.

17. Particular solutions to $(*)$ can be found by the method of undetermined coefficients. Suppose that there is a linear differential operator A such that $Af(x) = 0$. Then one can look for a particular solution to $(*)$ among the solutions to the homogeneous equation
$$ALy \;=\; 0.$$
by finding a suitable linear combination (i.e. determining suitable coefficients) of the linearly independent solutions to $ALy = 0$.

18. Particular solutions to $(*)$ can also be found by the method of variation of parameters. There is a particular solution of the form
$$y_p \;=\; c_1 y_1 + \cdots + c_n y_n$$
where the y_i are the linearly independent solutions to $(+)$ and the coefficients c_i are now functions of x. One obtains a system of simultaneous equations in $c_1' \ldots, c_n'$ whose coefficient matrix has determinant equal to the Wronskian $W(y_1, \ldots, y_n)$.

19. Simple mechanical vibration systems consisting of a mass m attached to a spring (with constant k) in the presence of friction (with constant c) and an external driving force $f(t)$ give rise to a second order linear ODE
$$mx'' + cx' + kx \;=\; f(t).$$

20. **Homogeneous systems.** An nth order homogeneous system is an equation of the form

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \qquad (+)$$

where $\mathbf{x} = \mathbf{x}(t)$ is an $n \times 1$ column vector of functions of t, and A is an $n \times n$ matrix whose entries are functions of t. If the entries of A are all constants, then the system is called constant coefficient.

A general solution of $(+)$ is of the form

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$$

where the c_i are constants (parameters) and the $\mathbf{x}_i(t)$ are n linearly independent column vectors.

21. **Linear systems.** A general linear system is an equation of the form

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{f}(t) \qquad (*)$$

where $\mathbf{x} = \mathbf{x}(t)$ is an $n \times 1$ column vector of functions of t, A is an $n \times n$ matrix whose entries are functions of t, and $\mathbf{f}(t)$ is a column vector of functions of t. If the entries of A are all constants, then the system is called constant coefficient.

The general solution of $(*)$ is of the form

$$\mathbf{x}(t) = \mathbf{x}_p(t) + \mathbf{x}_h(t)$$

where $\mathbf{x}_p(t)$ is a particular solution of $(*)$ and $\mathbf{x}_h(t)$ is the general solution of the associated homogeneous equation $(+)$.

22. **Eigenvalue-eigenvector method.** The solutions of $(+)$ in the constant-coefficient case are obtained by (1) solving $\det(A - \lambda I) = 0$ to find the eigenvalues λ of A; (2) for each eigenvalue λ solving the equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$ for eignevectors $\mathbf{v}$; and (3) constructing solutions of the form $e^{\lambda t}\mathbf{v}$.

One has to take care with (1) complex eigenvalues/eigenvectors, and with (2) repeated eigenvalues that may give rise to generalized eigenvectors and solutions of the form $e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2)$ etc.

The case of complex eigenvectors will be greatly simplified if you use Euler's identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

23. **Fundamental matrix and exponential matrix.** Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be n linearly independent solutions to $(+)$. Then writing these as column vectors of an $n \times n$ matrix gives the fundamental matrix $\Phi(t)$. Note that

$$\Phi(t)(\Phi(0))^{-1} = e^{At}$$

where the exponential matrix e^{At} is defined by the power series

$$e^{At} = I + At + \frac{A^2t^2}{2} + \cdots$$

For certain matrices A it is easier to directly compute e^{At} directly from the power series definition.

24. **Variation of Parameters.** Let $\Phi(t)$ be a fundamental matrix for the homogeneous system $(+)$. Then a particular solution of $(*)$ is obtained by the formula

$$\mathbf{x}_p(t) = \Phi(t)\int (\Phi(t))^{-1}\mathbf{f}(t)\,dt$$

25. **Laplace Transform.** The Laplace transform of $f(t)$ is a function $F(s) = \mathcal{L}\{f(t)\}$ defined by

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

26. **Properties of $\mathcal{L}\{f(t)\} = F(s)$.** The Laplace transform $\mathcal{L}\{f(t)\} = F(s)$ satisfies the following properties.

    (a) **Linearity.** $\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$ where a, b are constants and $G(s) = \mathcal{L}\{g(t)\}$.

    (b) **Power functions.** $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

    (c) **Exponential functions.** $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

    (d) **Sine.** $\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$

    (e) **Cosine.** $\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$

    (f) **Derivative (t–domain).** $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

    (g) **Derivative (s–domain).** $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$

    (h) **Shift (t–domain).** $\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$ where $u(t) = 0$ for all $t \leq 0$ and $u(t) = 1$ for all $t > 0$.

    (i) **Shift (s–domain).** $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

    (j) **Convolution.** $\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$ where $(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$.

27. **Laplace Transform applied to IVPs.**

$$\text{IVP} \xrightarrow{\mathcal{L}} \text{Algebra} \overset{\text{part. frac.}}{\longrightarrow} \text{Simpler algebra terms} \xrightarrow{\mathcal{L}^{-1}} \text{Solution to IVP}$$