Second Order Linear Equations.

The general form of a linear second order equation is

$$y'' + p(x)y' + q(x)y = f(x) \qquad (*)$$

This can be written as

$$Ly = f(x)$$

where L is the linear differential operator

$$L = D^2 + p(x)D + q(x).$$

1. **Existence/Uniqueness.** The first result tells us about existence and uniqueness of solutions to $(*)$. In order to talk about uniqueness we need to specify some *initial conditions*, such as the value of a solution and the value of its derivative at a particular input.

   Theorem. Suppose that the three functions $p(x)$, $q(x)$ and $f(x)$ are continuous on some open interval I in the real line, and suppose that a is a point of I. Then, given any pair of real numbers b_0 and b_1, the IVP

   $$y'' + p(x)y' + q(x)y = f(x) \qquad y(a) = b_0, \qquad y'(a) = b_1$$

   has a unique solution defined on all of I.

2. **Associated homogeneous equation.** Suppose that y_p is some particular solution to $(*)$ guaranteed by the previous result. Now if y is any other solution of $(*)$ then we can use linearity of L to say

   $$L(y - y_p) = Ly - Ly_p = f(x) - f(x) = 0.$$

   Thus $y - y_p$ is a solution of the *associated homogeneous equation*

   $$y'' + p(x)y' + q(x)y = 0 \qquad (+)$$

   We can summarize this as follows. The general solution to the ODE $(*)$ can be written as a sum

   $$y = y_p + y_h$$

   where y_p is a particular solution to $(*)$ and y_h is the general solution to $(+)$.

3. **Homogeneous equation — superposition principle.** Linearity helps us describe the general solution to the homogeneous equation $(+)$. If y_1 and y_2 are two solutions of $(+)$ and c_1 and c_2 are any two real numbers, then

   $$y = c_1y_1 + c_2y_2$$

   is also a solution to $(+)$.

4. **General solution of the homogeneous equation — theory.** We claim that there are two functions y_1 and y_2 with the property that the general solution of $(+)$ is of the form

   $$y = c_1y_1 + c_2y_2$$

   Indeed, given a point $a \in I$, the existence theorem implies that there is a function y_1 which is a solution of $(+)$ and which satisfies $y_1(a) = 1$ and $y_1'(a) = 0$.

   Another application of the existence theorem shows that there exists a function y_2 which is also a solution of $(+)$ and which satisfies $y_2(a) = 0$ and $y_2'(a) = 1$.

We claim that $c_1y_1 + c_2y_2$ is a general solution of (+). If y is an arbitrary solution of (+), then let $c_1 = y(a)$ and let $c_2 = y'(a)$. Now by the superposition principle $c_1y_1 + c_2y_2$ is a solution of (+), and by our choice of the constants c_i we have

$$c_1y_1(a) + c_2y_2(a) \;=\; (c_1)(1) + (c_2)(0) \;=\; c_1 \;=\; y(a)$$

and

$$c_1y_1'(a) + c_2y_2'(a) \;=\; (c_1)(0) + (c_2)(1) \;=\; c_2 \;=\; y'(a).$$

This means that y and $c_1y_1 + c_2y_2$ satisfy the same IVP, and by uniqueness we conclude that

$$y \;=\; c_1y_1 + c_2y_2 \qquad\qquad \text{on the interval } I.$$

Thus, an **arbitrary** solution y of (+) can be written as a linear combination $c_1y_1 + c_2y_2$ of y_1 and y_2. We say that $c_1y_1 + c_2y_2$ is a **general solution** of (+).

5. **Back to (∗).** Combining results from items 2 and 4 above we can write that the general solution to (∗) in the form

$$y \;=\; y_p + c_1y_1 + c_2y_2$$

where y_p is a particular solution to (∗) and $c_1y_1 + c_2y_2$ is the general solution to the associated homogeneous equation (+), which is written as a 2-parameter combination of the solutions y_1 and y_2.

6. **General solution of the homogeneous equation — practice.** In practice there are many other pairs of functions y_1 and y_2 that form the basis for a 2–parameter description of the general solution of (+). Such pairs are called **fundamental sets of solutions**.

We will give a proof of the fact that the following are three *equivalent characterizations* of fundamental sets later on. Let's get used to using them now.

The following three statements about solutions y_1 and y_2 of (+) are equivalent.

(a) The 2-parameter family

$$c_1y_1 + c_2y_2$$

is a general solution of (+) on I. That is, the collection $\{y_1, y_2\}$ is a fundamental set of solutions. Recall that this means that **every solution** y of (+) can be written as a linear combination $y = c_1y_1 + c_2y_2$ of y_1, y_2 on I.

(b) The **Wronskian** of the two solutions y_1 and y_2, defined by

$$W(y_1, y_2) \;=\; \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \;=\; y_1y_2' - y_1'y_2$$

is not zero at some input point a of I. That is

$$W(y_1, y_2)(a) \neq 0$$

Equivalently, $W(y_1, y_2) \neq 0$ at **all** points of I.

(c) The solutions y_1 and y_2 are **linearly independent on** I. This means that the only combination of

$$c_1y_1 + c_2y_2$$

which gives the constant function 0 on I is the trivial combination, $c_1 = 0 = c_2$. Equivalently, y_1 and y_2 are not scalar multiples of one another on I.

7. **Homogeneous equations with constant coefficients — finding solutions.** There is a cool strategy for finding solutions to (+) in the special case where $p(x) = b$ and $q(x) = c$ are constant functions.

   Strategy. Look for solutions of the form $y = e^{rx}$ for suitable numbers r. Since $y' = re^{rx}$ and $y'' = r^2 e^{rx}$, then (+) becomes

$$e^{rx}(r^2 + br + c) \;=\; 0$$

   But e^{rx} is never 0. Thus r satisfies the **characteristic equation**

$$r^2 + br + c \;=\; 0.$$

   This is a quadratic equation, and the quadratic formula will give solutions for r:

$$r \;=\; \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

   There are 3 possibilities.

   - **Distinct real roots.** ($b^2 - 4c > 0$) Then we get two solutions $e^{r_1 x}$ and $e^{r_2 x}$. It is easy to verify that they are linearly independent and therefore the general solution is

$$y \;=\; c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

   - **Repeated real root.** ($b^2 - 4c = 0$) In this case it is easy to verify that e^{rx} and xe^{rx} are solutions and that they are independent. In this case the general solution is

$$y \;=\; c_1 e^{rx} + c_2 x e^{rx}.$$

   - **Distinct complex roots.** ($b^2 - 4c < 0$) In this case r is a complex number, and the function e^{rx} is a complex-valued function (once we make sense of the exponential of a complex number). We will need to take some time aside to discuss complex numbers and complex-valued functions before dealing with this case.

Recognizing fundamental sets of solutions

–A proof that the three characterizations in item 6 above are equivalent–

We have seen from the existence and uniqueness theorem that every ODE of the form

$$y'' + p(x)y' + q(x)y = 0 \qquad (+)$$

where p, q are continuous on an open interval I has a fundamental set of solutions. For example, given a point $a \in I$, the solutions $\{y_1, y_2\}$ which satisfy

$$y_1(a) = 1, \qquad y_1'(a) = 0, \qquad y_2(a) = 0, \qquad y_2'(a) = 1$$

forms a fundamental set.

However, in practice (for example if you find two solutions by solving a characteristic equation) you may be given solutions y_1, y_2 to (+) on I which do not satisfy the nice 1–0–0–1 pattern above. How can you tell if $\{y_1, y_2\}$ forms a fundamental set? Well, item 6 above provides two alternative (equivalent) characterizations of when two functions $\{y_1, y_2\}$ form a fundamental set of solutions (a) of the ODE (+). There is characterization (b) which requires that the Wronskian $W(y_1, y_2) \neq 0$ on I, and characterization (c) which requires that the solutions be linearly independent on I.

Conditions (a) and (b) of item 6 are equivalent. Well, the key fact that made the set $\{y_1, y_2\}$ with the 1–0–0–1 pattern above work is that it was easy to solve the initial condition equations

$$c_1 y_1(a) + c_2 y_2(a) = b_0$$

$$c_1 y_1'(a) + c_2 y_2'(a) = b_1$$

for an arbitrary RHS b_0, b_1. So all we have to check is that at some point a the equations above have a unique solution for all possible choices of RHS. Using the language of column vectors these equations become one vector equation.

$$c_1 \begin{pmatrix} y_1(a) \\ y_1'(a) \end{pmatrix} + c_2 \begin{pmatrix} y_2(a) \\ y_2'(a) \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \qquad (A)$$

This equation will have a unique solution for every RHS precisely when the two vectors $\begin{pmatrix} y_1(a) \\ y_1'(a) \end{pmatrix}$ and $\begin{pmatrix} y_2(a) \\ y_2'(a) \end{pmatrix}$ are linearly independent vectors in $\mathbb{R}^2$. This means that neither vector is a multiple of the other. A simple algebra way to formulate this is that

$$y_1(a)y_2'(a) - y_1'(a)y_2(a) \neq 0$$

If you brought all y_1 terms to one side and all y_2 terms to the other, this equation would become the equation stating that the slopes of the vectors $\begin{pmatrix} y_1(a) \\ y_1'(a) \end{pmatrix}$ and $\begin{pmatrix} y_2(a) \\ y_2'(a) \end{pmatrix}$ are the same. But since some of the denominators could be 0, it is better to write this equality of slopes out as above.

Determinants. The expression $ad - bc$ is called the *determinant* of the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and is denoted by

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

and it is 0 if and only if one of the two column vectors $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$ is a scalar multiple of the other.

Wronskians. Given two differentiable functions y_1 and y_2 their *Wronskian* is defined to be the following determinant

$$W(y_1, y_2) \;=\; \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \;=\; y_1 y_2' - y_1' y_2$$

It is a function of x. Condition (a) above that $\{y_1, y_2\}$ is a fundamental set of solutions to (+) is equivalent to the condition that $W(y_1, y_2)(a) \neq 0$ for some input point $a \in I$.

The argument above shows that conditions (a) and (b) of item 6 are equivalent.

Well there is still the question of why $W(y_1, y_2)(a) \neq 0$ implies that $W \neq 0$ for all other points of I. This follows from the following lovely formula due to Abel. We write $W(x)$ as short hand for the Wronskian $W(y_1, y_2)(x)$ considered as a function of x.

$$W(x) \;=\; W(a) e^{-\int_a^x p(s)\, ds}$$

where $p(x)$ is the coefficient function from the linear homogeneous ODE (+). Note that the exponential is never 0, thus $W(x)$ is never 0 on I if it is not 0 at some point $a \in I$. This completes the statement of (b) in item 6.

Abel's formula is proven by first showing that $W(x)$ satisfies a separable ODE

$$\frac{dW}{dx} + p(x) W \;=\; 0$$

The latter is seen by differentiating the terms in W using the product rule, and then using the fact that y_1, y_2 satisfy the ODE (+) to see that the LHS of the equation above all works out to be 0.

$$\frac{dW}{dx} + pW \;=\; (y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2') + p(y_1 y_2' - y_1' y_2)$$

Note that the $y_1' y_2'$ terms with opposite signs cancel, and we factor y_1 and y_2 out of the remaining 4 terms to get

$$\frac{dW}{dx} + p(x) W \;=\; y_1(y_2'' + p y_2') - y_2(y_1'' + p y_1') \;=\; y_1(-q y_2) - y_2(-q y_1) \;=\; 0$$

where the second from last equality holds because y_1, y_2 are solutions to (+).

Conditions (b) and (c) of item 6 are equivalent. This is the same as proving that the negation of (b) and the negation of (c) are equivalent.

First, we show that the negation of (c) implies the negation of (b). Now the negation of (c) means that y_1, y_2 are linearly dependent on I. This means that there exist numbers c_1, c_2, not both zero, such that

$$c_1 y_1 + c_2 y_2 \;=\; 0$$

on I. Differentiating gives

$$c_1 y_1' + c_2 y_2' \;=\; 0$$

on I. Evaluating at some point $a \in I$ this means that the simultaneous equations

$$c_1 y_1(a) + c_2 y_2(a) \;=\; 0$$

$$c_1 y_1'(a) + c_2 y_2'(a) \;=\; 0$$

have a non-zero solution c_1, c_2. This can only happen when $W(y_1, y_2)(a) = 0$. This is the negation of condition (b).

Finally, we argue that the negation of condition (b) implies the negation of condition (c). We really use the fact that y_1, y_2 are solutions of (+) here. If the negation of (b) holds, then there is a point $a \in I$ such that $W(y_1, y_2)(a) = 0$. This means that there is a non-trivial solution (c_1, c_2 not both zero) to the equations

$$c_1 y_1(a) + c_2 y_2(a) \;=\; 0$$

and

$$c_1 y_1'(a) + c_2 y_2'(a) \;=\; 0$$

This means that the combination $y = c_1 y_1 + c_2 y_2$ satisfies (+) and $y(a) = 0$ and $y'(a) = 0$. But the constant function $y = 0$ is also a solution of (+) which satisfies these initial conditions. By uniqueness, $c_1 y_1 + c_2 y_2 = 0$, and so we have shown that y_1, y_2 are linearly dependent on I. Thus the negation of (b) implies the negation of (c).

Remarks. Some of the arguments above really used the fact that y_1, y_2 were solutions to a homogeneous linear second order ODE. The facts proven in item 6 are not true for general functions.

- The argument about Wronskians not vanishing at a point being equivalent to them not vanishing at all points of I is one place where we used (+).

  Here is an example of two simple functions x^2 and x which have Wronskian

  $$W(x^2, x) \;=\; \det \begin{pmatrix} x^2 & x \\ 2x & 1 \end{pmatrix} \;=\; x^2 - 2x^2 \;=\; -x^2$$

  Note that $W = 0$ at input 0, but that $W \neq 0$ at all other real number inputs. This means that x^2 and x are not both solutions of the same homogeneous linear second order ODE defined on all of $\mathbb{R}$.

- The argument about linearly independent solutions having non-zero Wronskian is another place where we invoked (+) (through the uniqueness result).

  The functions $f(x) = x^2$ and $g(x) = x|x|$ are easily checked to be linearly independent on $\mathbb{R}$. Both are differentiable functions on $\mathbb{R}$ and have derivatives (check this carefully!) $f'(x) = 2x$ and $g'(x) = 2|x|$. Thus their Wronskian is

  $$W(f, g) \;=\; \det \begin{pmatrix} x^2 & x|x| \\ 2x & 2|x| \end{pmatrix} \;=\; 2x^2|x| - 2x^2|x| \;=\; 0$$

  is identically 0 on all of $\mathbb{R}$. Again, this means that f and g are not solutions to a second order linear homogeneous ODE defined on all of $\mathbb{R}$.

Summary of Results for an nth order linear ODE

1. **General form of the nth order linear ODE.**

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y \;=\; f(x) \qquad (*)$$

2. **The associated nth order linear homogeneous ODE.**

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y \;=\; 0 \qquad (+)$$

3. **Existence and uniqueness of solutions.** Suppose that $p_1(x), \ldots, p_n(x)$ and $f(x)$ are all continuous on an open interval I containing a point a. Given any choice of real numbers $b_0, \ldots, b_{n-1}$ there exists a unique solution to $(*)$ satisfying the initial conditions

$$y(a) = b_0, \quad y'(a) = b_1, \quad \ldots \quad , \quad y^{(n-1)}(a) = b_{n-1}.$$

4. **General form of solution to $(*)$.** The general solution to $(*)$ can be written as

$$y \;=\; y_p + y_h$$

where y_p is a particular solution to $(*)$ and y_h is the general solution to $(+)$.

5. **General solution to $(+)$.** The general solution to $(+)$ can be written in the form

$$y_h \;=\; c_1 y_1 + \cdots + c_n y_n$$

where c_i are constants (real numbers) and $\{y_1, \ldots, y_n\}$ is a set of n linearly independent solutions to $(+)$.

6. **Linear Independence.** The collection $\{y_1, \ldots, y_n\}$ is a set of n linearly independent functions on the interval I if the only solution to

$$c_1 y_1 + \cdots + c_n y_n \;=\; 0$$

on I is the trivial solution $c_1 = 0, \ldots, c_n = 0$. That is the only linear combination of the y_i which yields the constant function 0 on I is the trivial linear combination where all coefficients are 0.

7. **Linear Independence and Wronskians.** In the case $y_1, \ldots, y_n$ are the solutions of the ODE $(+)$ on I, then the linear independence condition can be rephrased as

$$W(y_1, \ldots, y_n)(a) \;\neq\; 0 \qquad \text{for some } a \in I.$$

and equivalently (by a version of Abel's formula) $W(y_1, \ldots, y_n)$ is never 0 on I.

Here $W(y_1, \ldots, y_n)$ denotes the Wronskian of $y_1, \ldots, y_n$ and is defined to be the following determinant

$$W(y_1, \ldots, y_n) \;=\; \det \begin{pmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & & \vdots \\ y_n^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$